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A Simplified Approach To The Yawing Motion Of A Spinning Shell

SERGE J. ZAROODNY

DEPARTMENT OF THE ARMY PROJECT No. 503-03-001
ORDNANCE RESEARCH AND DEVELOPMENT PROJECT No. TB3-0108K

BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

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A B E R D E E N   P R O V I N G   G R O U N D,   M A R Y L A N D

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PREFACE

This paper constitutes a departure from the previous, formal and analytical, treatments of the subject. It presents little that is novel; it is written colloquially; it is concerned just as much with mere verbal qualitative discussions as with quantitative evaluations; it is frankly concerned more with reader's attitude to the facts than with the facts themselves; and it makes no pretense of brevity. Nevertheless, this presentation has been found necessary. There are many concepts which ought to be more widely understood if our thinking is not to be ossified, and if we are to continue - by working together - to push out the frontiers of ballistics. In the present state of art not all of these concepts can be put in the precise language of mathematics; but to continue to ignore them would be a poor substitute for precision. Verbal discussion and studied colloquialism have a legitimate - if modest - place in ballistics. Certainly, facts are of no use unless they are known to those who have an opportunity to use them. Brevity is often overdone: a ballistician spends many of his evenings with pencil and paper, trying to understand the unstated implications of a brief - and therefore allegedly easy - text.

The premise of this paper is that a visualization of a physical phenomenon is useful: not so much because on occasions it might be all that is necessary, as because on few occasions it might open a new vista. Of course, visualization is part-and-parcel of any understanding; but the reader knows himself that it is all too often crowded out by algebra.

The paper is meant to supplement, rather than to supplant, the existing texts on the subject. Thus, the avoidance of the calculus is, of course, merely a pose; but it is a pose which does not harm, and which might - on occasions - be stimulating.

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SJZaroodny/plg
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ABSTRACT

An attempt is made to free the theory of yawing motion of a shell from the excessive and frustrating reliance upon the assumptions of linearity. A method of looking upon the yawing motion is suggested whereby the physical significance, or the mechanism, of the effect produced by various forces and torques acting upon the shell can be vividly visualized, without resorting to the linearization of the problem (or to calculus).

The method consists essentially of considering the vector of angular momentum L, and separating the dynamic concept (the motion of L under the influence of applied torque, here termed "quasi-precession") from the kinematic concept (the motion of the shell with respect to L, here termed "quasi-nutation").

The method is tested by an application to the solved, linear, case: it is used to derive - and to interpret - all conclusions of the linear theory.

The principal value of the method is pedagogical. It shows a promise in tackling certain more complicated problems of ballistics, such as stability of liquid-filled shell.

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LIST OF SYMBOLS

A_1	Unit vector along shell axis; A = axial moment of inertia
B	Moment of inertia about a transverse axis through the c.g. and perpendicular to A.
C_n, C_p	Complex numbers such that $C_p + C_n = 1$.
d	Diameter of shell.
f	A function of quasi-nutation and yaw.
F	Magnus force.
g	Acceleration due to gravity.
G	Vector of the sum of all applied torques.
H	Damping torque.
L	Vector of angular momentum.
m	Mass of body.
M	Overturning torque.
N	The normal force.
o	The c.g.
QN	Quasi-nutation (the angle A_1oL).
QP	Quasi-precession (the angle LoT_1).
S	Synge's force.
s	Linear Stability factor.
T_1	Unit vector along trajectory; T = magnus torque
u	Velocity of c.g.
β	An auxiliary angle as shown on sketches.
γ	An auxiliary angle as shown on sketches.
δ	The yaw.
Δ	The quasi-precession.
η	Component of spin (ω) perpendicular to longitudinal axis of body.
μ	Coefficient associated with M, defined by $M = \mu \sin\delta$ or $M = \mu\delta$.
ξ	Yaw (viewed as a complex number in the linear theory).
$\dot{\xi}$	The rate of change of yaw.
ρ	Density of the air.
σ	$= \sqrt{1-1/s}$
ϕ	Quasi-nutation.

$\underline{\omega}$	Vector of angular velocity of shell.
ω_1	Spin
Ω	Angular velocity of quasi-nutation with respect to instantaneous position of angular momentum.
$\mathcal{A}$	Axial torque
$\mathcal{L}$	Lift
ν	Spin in radians/caliber of travel.
XT	Cross-Magnus torque.
XF	Cross-Magnus force
h	$H/\eta = K_H \rho d^4 u$
t	$T/\delta = K_T \rho d^4 u\omega$
ℓ	$\mathcal{L}/\delta = K_L \rho d^2 u^2$
J_i	$K_i \rho d^3/m$
θ	Inclination of trajectory
J_θ	$gd\sin\theta/u^2$

INTRODUCTION

The theory of the yawing motion of a shell is an ancient and important subject of ballistics: it lies at the root of well-nigh all problems of ordnance that involve the accuracy of fire. It seems that this theory deserves being known more widely, and used more frequently, than is the case now. Also, it deserves an occasional re-assessment.

Traditionally, the subject has been the domain - and perhaps, monopoly - of what we shall call the mathematical approach. This situation, apparently, is the result of the elegance and power with which this approach solves the basic problems of the stability of a shell. We now take it for granted that the first step in any investigation of the yawing motion must be an idealization of the problem; that the elementary physical facts must be condensed, in a simplified form, into the mathematical notation; that the interplays of these facts can be handled as mere algebraic substitutions, and the totality of these facts must be expressed as a system of differential equations, the solution of which may be obtained by abstract mathematical methods, not requiring an initmate understanding of the physical significance of each intermediate step; that the physical significance, if and when needed, will re-emerge in examining the mathematical solution; that the experimental tests must be designed, and the experimental data examined, around this abstract solution; and finally, that predictions may be made by these methods that will possess considerable generality.

It is of interest to observe that this tradition appears to be relatively young; for it is difficult to trace it far beyond the classical work of Fowler in the First World War. Raised in the belief in the power of mathematical deduction, we might surmise, for instance, that the introduction of rifling was a result of Euler's theory of spinning top; this indeed might have been relevant as far as artillery is concerned. But the rifles of the American Revolution were made by the village blacksmith, innocent of the theory, while the most advanced nation in Europe was still mass-producing the smooth-bore Brown Bess. The importance of the mathematical approach, clearly, rests not upon tradition, but on hopes that it will produce useful results sooner than a haphazard invention and empiricism can produce them. It is therefore not inappropriate that from the proving-ground point of view ("the proof of the pudding is in the eating") we inquire, occasionally, whether the mathematical approach itself - that is, the idea that it is possible to apply the rigor of mathematics to a field of engineering (to wit, ballistics) - has lived up to the expectations.

The fact is, the problem of the yawing motion of a shell has been solved (and indeed, vindicated), only for the highly simplified, linear case. Since the underlying assumption usually is that the yaw is small, this is as though the theory solves the problem only when the shell flies rather satisfactorily.

When some of the simplifying assumptions (such as the linearity of the force and torque system, smallness of the yaw, symmetry of the shell, etc.) have to be rescinded, the complexity of the problem grows rapidly. In fact, most of the present work in this field anticipates an extensive use of modern high-speed computing machines. In this connection it is particularly

annoying that with the recision of the linearity it is no longer legitimate to utilize that basic feature of the linear theory, the decomposition of the yaw into two normal modes, precession and nutation. The numerical solution of the equations of yawing motion becomes extremely laborious, and yet much of the information expected from this solution (e.g., the information that the shell is yawing approximately epicycloidally) resembles very much the information already available from the linear theory. The practical result usually is that the available indications of non-linearity are simply suppressed - rather than investigated. In other words (to mix a metaphor), the elegance of the linear theory has exerted such influence on the methods of ballistics that the present-day theory is chained to the assumptions of linearity as to a point of diminishing returns. There exists an acute need for a more powerful method of tackling the problem of yawing motion.

The method suggested hereby has no claim of being a complete answer to this need; yet it might be a contributing factor. What it has to offer is simply an easier visualization of the problem; and it is quite possible that a future method will be dependent upon a visualization and a judgment.

In one problem, particularly - that of liquid-filled shell - the traditional mathematical approach appears to be simply stalemated. It has not been possible to get even to first base, i.e., to formulate total differential equations of the motion; while a solution of partial differential equations of the motion of the liquid within the shell (with the object of using the results in determining the motion of the shell) seems a long way off. It is in connection with this problem that the method suggested in this report arose. It started as an attempt to by-pass the differential equations by means of visualizing the motion of the liquid within the yawing shell, or as an esthetic, or qualitative "reasoning" (if such a term may be applied). It was not easy, however, to meet the contemporary standards set by the tradition of the mathematical approach; in fact, a number of pitfalls of this "qualitative reasoning" have been found. For this reason it appeared best to expound this method for the simple case of a rigid shell, and to test it by applying it to the solved linear case (where this method can be used to re-derive and to interpret all conclusions of the linear theory); postponing the possible application of this method to the cases of non-linearity, asymmetry, and liquid filler.

PRELIMINARIES

Our method is non-mathematical only in this respect; we shall not require of the reader the formal knowledge of calculus and of differential equations. This is rather a tenuous simplification, since the differential equations, of course, are the essence of the problem. On the other hand, we shall expect the reader to have a mastery of algebra, geometry, plane (and possibly, spherical) trigonometry; of the concepts of torque and angular velocity as vectors defined in the right-handed sense; of the principal axes of inertia (i.e., the understanding that a rigid body may spin steadily only if it is dynamically balanced about the axis, and that there are at least three such, mutually perpendicular, axes in every rigid

body); of the moments of inertia; of the concept of the vector of angular momentum (which can be constructed by decomposing the instantaneous vector of angular velocity along the three mutually perpendicular axes of inertia of the body, multiplying each component by the corresponding moment of inertia, and recombining the products vectorially); of the fact that the vector of applied torque is the velocity of the tip of the vector of angular momentum; of the resolution of the motion of a rigid body into the motion of its center of gravity and the rotation about this center of gravity; and of the use of complex numbers.

We shall consider only an axially-symmetric, spindle-like body with moments of inertia A and B ($A < B$, B the same about every transverse axis through the center of gravity). For such a body the unit vector along the axis ($\underline{A}_1$), the vector of angular velocity ($\underline{\omega}$) and the vector of angular momentum ($\underline{L}$) are always co-planar (Figure 1). For, if the instantaneous motion of the body (with respect to its center of gravity) is a rotation at an angular velocity ω about an axis ($\underline{\omega}$) that is different from $\underline{A}_1$, the vector $\underline{\omega}$ can be resolved into the axial spin ω_1 along $\underline{A}_1$ and the cross-spin η perpendicular to $\underline{A}_1$; because of the symmetry, the direction $\underline{\eta}$ of η may be taken as the second principal axis of inertia (there being no component of $\underline{\omega}$ along the third axis), and $\underline{L}$ can be constructed from the component $A\omega_1$ along $\underline{A}_1$ and $B\eta$ along $\underline{\eta}$, i.e., in the plane of $\underline{A}_1$ and $\underline{\omega}$. In particular, since $B > A$, $\underline{L}$ is always further away from $\underline{A}_1$ than $\underline{\omega}$ is, and there exists the relation

$$\frac{\tan(\underline{A}_1, \underline{\omega})}{\tan(\underline{A}_1, \underline{L})} = \frac{\eta / \omega_1}{B\eta / A\omega_1} = A/B \qquad (1)$$

Note that this relation allows a coincidence of the vectors $\underline{\omega}$ and $\underline{L}$ (which is to say, allows a steady unrestrained rotation of the body, or a steady rotation with no torque applied) only when the angle between $\underline{A}_1$ and $\underline{\omega}$ is 0, π or $\pi/2$; in which cases the motion is either a pure spin (right-handed or left-handed) or a pure cartwheeling.

Generally, however, the vector of angular velocity $\underline{\omega}$ is fixed neither in space nor in the body. It is for this reason, perhaps, that students of ballistics have some slight difficulties in visualizing the three-dimensional yawing motion of a rigid body.

The visualization is facilitated by the decomposition of the vector $\underline{\omega}$ into the components ω_1 and η. The cross-spin η, of course, is simply the velocity of the tip of the unit vector $\underline{A}_1$ in the surface of the sphere which is centered at the c.g. of the body, and moves with the body, but does not rotate; this velocity, of course, is at right angles to the vector $\underline{\eta}$. In this way we replace the visualization of the motion of the body (rotation about $\underline{\omega}$) by a visualization of the motion of a line $\underline{A}_1$, with the

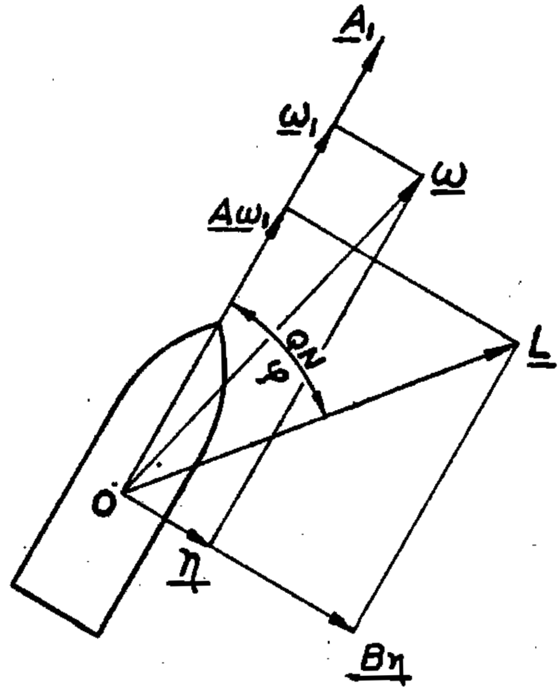


Fig. 1: Angular Velocity & Angular Momentum

axial spin ω_1 superposed if and when necessary. In other words, a three-degree-of-freedom situation is replaced by a two-dimensional situation (in the surface of the unit sphere). The motion of the tip of the unit vector $\underline{A}_1$ in the surface of the unit sphere can be specified, of course, as a rotation about the vector $\underline{\omega}$.

However, the vector $\underline{\omega}$ is not a convenient point of reference. The vector $\underline{L}$ is much more convenient. We shall presently see that it does not move as fast as $\underline{\omega}$ and in particular, we know how it does move - viz., the velocity of its tip is the vector $\underline{G}$ of the applied torque. The motion of the tip of $\underline{A}_1$ may be described as a rotation about the instantaneous position of $\underline{L}$ at an angular velocity

$$\Omega = \frac{\eta}{\sin(\underline{A}_1, \underline{L})} = \frac{\eta}{B\eta/L} = L/B, \qquad (2)$$

where L is the magnitude of $\underline{L}$.

We can thus describe the motion of the body by two statements:

I. $\underline{A}_1$ rotates about the instantaneous position of $\underline{L}$ at an angular velocity Ω, and

II. The velocity of the tip of $\underline{L}$ is $\underline{G}$,

it being understood that $\underline{G}$ can be determined from the specification of $\underline{A}_1$ and $\underline{L}$ (and other relevant and available data). The statement II, of course, is simply the applicable law of physics; statement I connects the easily visualizable (and measurable) vector $\underline{A}_1$ with the very relevant physical quantity, $\underline{L}$. In most texts on ballistics the vector of angular momentum is mentioned briefly in the beginning of a derivation, and thereafter is replaced by the appropriate combinations of its elements (A, B, ω_1 and η). Our suggestion amounts to inviting the reader to "think in terms of angular momentum", i.e., to make a better - more frequent, and more intimate - use of this essential concept. The two statements, together with a specification of $\underline{G}$ correspond, of course, to the differential equations of the angular motion of the shell; they are to be augmented by the statement that the acceleration of the center of gravity is the applied force divided by the mass (with force properly specified).

VACUUM NUTATION

An important special case occurs when there are no torques acting upon the body (e.g., shell flies in vacuum). The the vector $\underline{L}$ is constant. A very simple description of the motion is then possible. As the vector $\underline{A}_1$ on Figure 1 comes out of the plane of the paper (rotating about $\underline{\omega}$), the co-planarity of $\underline{A}_1$, $\underline{\omega}$ and $\underline{L}$ must be preserved; hence the

vector $\underline{\omega}$ comes out of the plane of the paper, too. Since the angle $\underline{A}_1 O \underline{L}$ is not changing, by (1) the angles $\underline{A}_1 O \underline{\omega}$ and $\underline{\omega} O \underline{L}$ are not changing either; hence the whole triplet of vectors $\underline{A}_1$, $\underline{\omega}$ and $\underline{L}$ is rotating (as a rigid plane) about $\underline{L}$. The motion of the body is a rolling of the circular cone of half-angle $\underline{A}_1 O \underline{\omega}$ on the circular cone of half-angle $\underline{\omega} O \underline{L}$.

It is not amiss at this point to make several trivial observations. The motion is not a simple rotation, it is rolling; yet it is a basic, steady, type of motion, that may go on forever - if our assumptions (so far, only the rigidity and symmetry of the body and the absence of applied torque) hold; this is to say that the motion can be predicted for any length of time. If the assumptions are not quite exact, the motion still can be predicted, with some accuracy, for some interval of time - the more nearly true these assumptions, the greater this interval. The angle $\underline{A}_1 O \underline{L}$ may have any value from 0 to π; it may also have an arbitrary initial orientation (i.e., there are two scalars representing the arbitrary constants of integration).

We shall call the angle $\underline{A}_1 O \underline{L}$ "quasi-nutation".

Since in this problem the only direction fixed in space is $\underline{L}$, viz., we may visualize the shell's c. g. as stationary in space, it might be also natural to call this angle - in this case - the yaw. Thus the quasi-nutation could be defined as the yaw with respect to $\underline{L}$; or, we might say that in vacuum nutation the yaw with respect to $\underline{L}$ consists of quasi-nutation.

OVERTURNING MOMENT

The torque (or moment) acting upon the shell in flight is specified, conventionally, as the sum of its components; these components, in turn, being specified both in direction and magnitude, in certain convenient ways. It is not quite a simple matter to outline, at the outset, which components (or, as the parlance goes, which torques) should be considered; the student of ballistics would find it best to go by easy steps, digesting well the basic concepts, and accepting or rejecting the more advanced concepts as the need for them, as well as the practicability of handling them, becomes clear.

Of these torques the most important one, theoretically, is the one whose vector is perpendicular to the plane of yaw (i.e., whose couple is in the plane of yaw), and whose magnitude depends essentially upon the angle of yaw (angle from trajectory of center of gravity to axis of shell). It is called usually simply "moment", since historically it was the only moment at first considered; and, since it was first considered in connection with spin-stabilization of artillery shell, it is reckoned, conventionally, positive when overturning. Hence the more exact name (distinguishing it from other components) is overturning torque. With fin-stabilized projectiles the righting torque is reckoned merely as a negative overturning torque.

In the linear theory the magnitude of this torque (M) is assumed to be proportional to the angle of yaw (δ). The solution then is remarkably elegant. In certain texts on ballistics M is assumed to be (for the reasons of certain convenience in the preliminary mathematics) proportional to $\sin\delta$ (often a misleading assumption) or $\tan\delta$ (sometimes a better approximation, though still limited to small angles of yaw); in practice, both of these assumptions eventually revert to the assumption of linearity in δ. The exact (and often not simple) function $M(\delta)$ can be obtained experimentally, say, by wind-tunnel tests. An exact mathematical solution has been worked out but is - in comparison with the linear theory - so cumbersome as to be considered impractical. It is precisely this gap between the linear and non-linear theories which our approach attempts to bridge (or more exactly, our approach attempts to outline a method of bridging this gap).

NUTATION AND PRECESSION

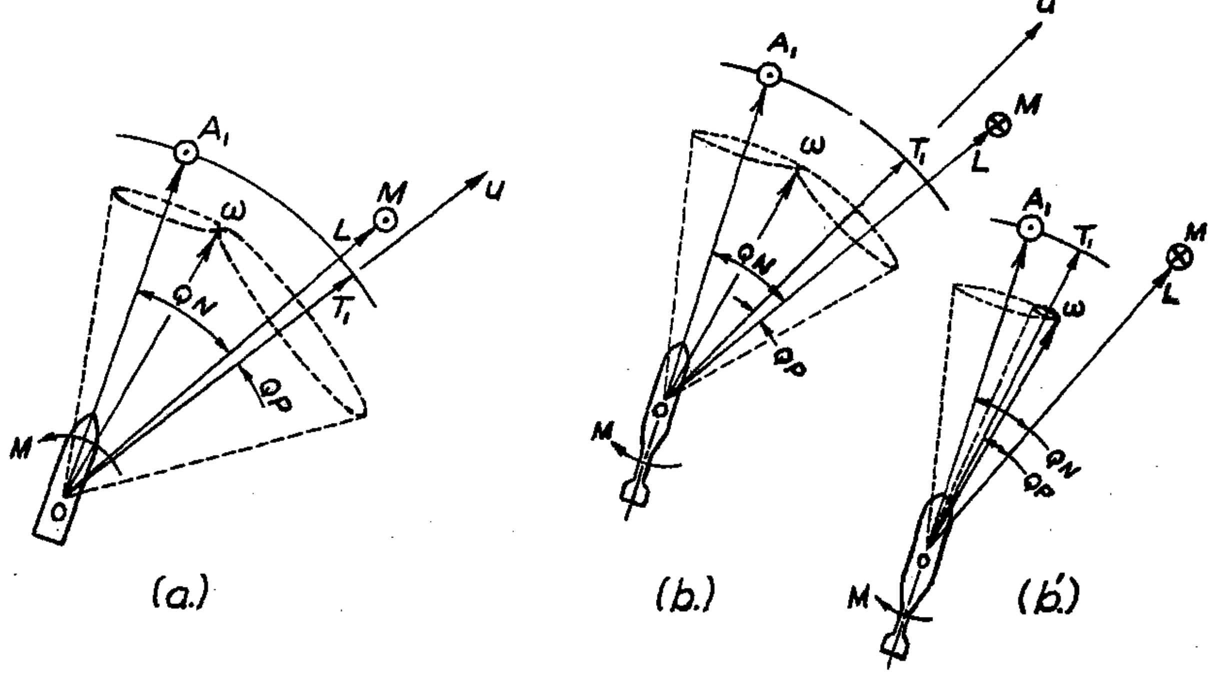
There are two special cases when the yawing motion is particularly simple - to wit, circular - even with a non-linear $M(\delta)$. These cases arise particularly naturally if we assume, for the time being, that $\underline{M}$ is very small, i.e., $\underline{L}$ moves only slowly; the motion is almost, but not quite, the same as the nutation in vacuum.

Let $\underline{T}_1$ be a unit vector in the direction of the trajectory, i.e., in the direction of the velocity $\underline{u}$ of the center of gravity of the shell.

Consider first the case when $\underline{T}_1$ is approximately in the same direction as $\underline{L}$. The situation may be such (Figures 2a and 2b) that the triplet of vectors $\underline{A}_1$, $\underline{\omega}$ and $\underline{L}$ rotates, as a rigid plane, about $\underline{T}_1$. In particular, if $\underline{M}$ is an overturning torque (Figure 2a), $\underline{L}$ would be coming out of the plane of the paper; the motion then may be steady if $\underline{T}_1$ is farther from $\underline{A}_1$ than $\underline{L}$ is. Or, if $\underline{M}$ is a righting torque (Figure 2b), $\underline{L}$ would be moving into the plane of the paper; but the motion still may be steady if $\underline{T}_1$ is closer to $\underline{A}_1$ than $\underline{L}$ is. In either case the motion of the body is a rolling of the cone $A_1O\omega$ (same as before), but this time not on the cone of half-angle ωOL (as was the case in vacuum) - rather, on the cone of half-angle ωOT_1. The angular velocity of the plane of yaw is slightly lower than Ω with spin-stabilized shell (Figure 2a), slightly higher than Ω with fin-stabilized shell (Figure 2b).

Such a motion is call nutation. In it the yaw (the angle A_1OT_1) is approximately the same as quasi-nutation (angle A_1OL), but also has a slight component T_1OL. We shall call the angle T_1OL "quasi-precession". Note that with both spin-stabilized and fin-stabilized shell the nutation is in the same direction as the spin.

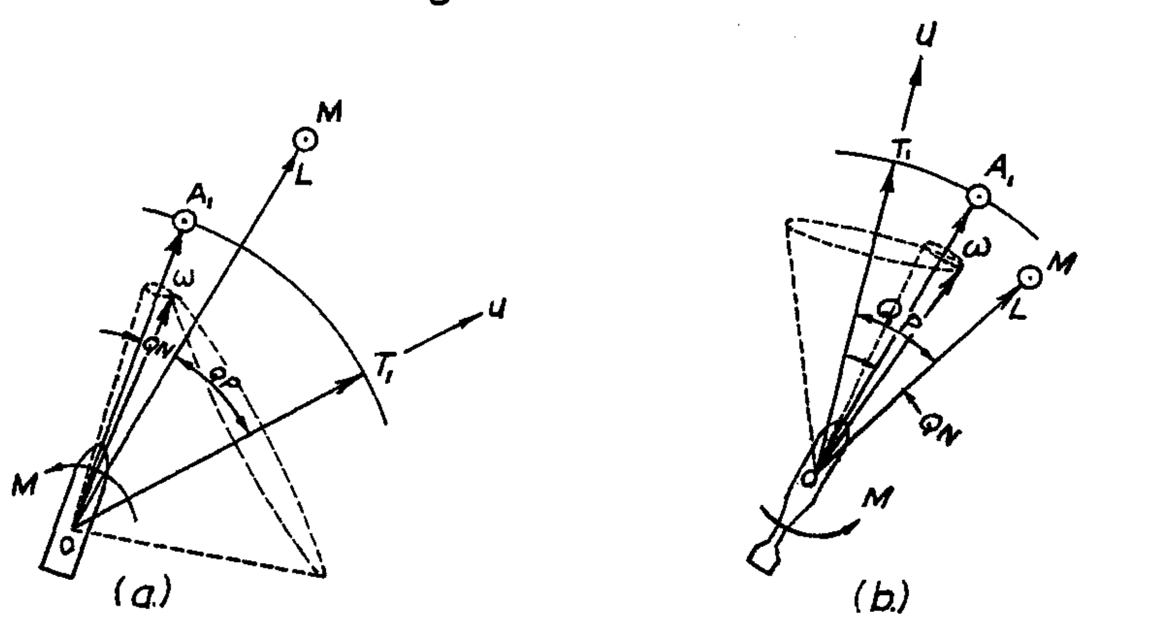
Notation: ⊙ *denotes the tip of arrow coming out of the paper*
⊗ *denotes the tail (feathers) of arrow going into the paper*



spin-stabilized shell

fin-stabilized shell
(T_1 between A_1 & L)

Fig. 2: Nutation



spin-stabilized shell

fin-stabilized shell
(T_1 outside of angle A_1OL; left-handed yawing)

Fig. 3: Precession

Consider next the case when $\underline{T}_1$ is far away from the triplet of vectors $\underline{A}_1$, $\underline{\omega}$ and $\underline{L}$. This might be the case, for instance, when the angle A_1OL is small; and this would mean that the cross-spin η is small, i.e., the shell axis $\underline{A}_1$ is moving only slowly. The situation again may be such (Figures 3a and 3b) that the triplet $\underline{A}_1$, $\underline{\omega}$ and $\underline{L}$ rotates as a rigid plane about $\underline{T}_1$. In particular, if $\underline{M}$ is an overturning torque, $\underline{T}_1$ is still on the same side of $\underline{A}_1$ as $\underline{L}$ is, though much farther away (Figure 3a). Or, if $\underline{M}$ is a righting torque, $\underline{T}_1$ must be on the other side of $\underline{A}_1$ from $\underline{L}$ (Figure 3b). The motion, again, is a rolling of the same cone $A_1O\omega$ on the cone ωOT_1. For spin-stabilized shell the angular velocity of the plane of yaw is in the same direction as spin, but much slower than the rate of vacuum nutation; for fin-stabilized shell, however, the motion is in a direction opposite to spin.

Such a motion is called precession. In it the yaw consists mostly of the quasi-precession, but there is also a slight component of quasi-nutation.

Nutation and precession are obviously pretty much the same kind of phenomenon; the difference between them appears to be merely quantitative. In fact, this seems to be precisely the case as far as the original, astronomical, usage of these words is concerned; the phenomena which go by these names are of the same nature, viz., the substantially circular yawing motion of an axis of a spinning symmetrical rigid body under the influence of certain external torques. Yet it would not be right to ignore the sense of the qualitative distinction, conveyed by the very choice of the words: precession, coming from the same root as "precede", "proceed", "procession", emphasizes the stateliness, viz., steadiness and slowness, of the motion; while nutation (meaning nodding) emphasizes the disturbance, the unsteadiness, - i.e., perturbation, or oscillation. Thus, while we now find it convenient (from the mathematical point of view) to consider these two phenomena separately, and as being on a par, it is essential to keep in mind that historically they arose as components of a larger, more complicated, motion; and that the precession is somehow more fundamental, the nutation being of the nature of a refinement on the precession.

In our case the qualitative distinction is that the nutation emphasizes the motion of shell axis about the vector of angular momentum (so to say, a kinematic concept), while precession emphasizes the motion of the vector of angular momentum about some other axis, in this case, trajectory (a dynamic concept). Our concepts of "quasi-nutation" and "quasi-precession" serve further to crystallize this qualitative distinction.*

*These terms should be acceptable to purists: "quasi" means "something like", and besides, like "nutation" and "precession", is Latin; while "pseudo" means "falsely appearing as", and is Greek. Yet the terms are still rather vague: quasi-nutation is really meant to be the essence of nutation, and quasi-precession, the essence of precession.

While the nutation and the precession, so defined, do not depend on the assumption of linearity of $M(\delta)$, neither do they give us much help in passing from these two special cases to the general (epicyclic) case of yawing motion.* Their importance, however, is very much enhanced once the assumption of linearity of $M(\delta)$, i.e., the smallness of δ, is made. Then their angular rates (and also, as the more elaborate theory shows, their rates of damping) become independent of the angle of yaw, and these two types of motion become superposable. Moreover, they then become "normal modes" of motion (viz., not only independent and superposable, but also particularly simple modes). In fact, the elegance of the mathematical approach amounts to this: in the linear theory it is possible to decompose any complicated (epicyclic) motion in two simple components, nutation and precession.

We decompose the yaw (the angle from $\underline{T}_1$ to $\underline{A}_1$) in an alternate way, viz., into quasi-precession (the angle from $\underline{T}_1$ to $\underline{L}$) and quasi-nutation (the angle from $\underline{L}$ to $\underline{A}_1$), essentially for the reason that these concepts survive in any general, non-linear, theory: i.e., we simply bring in the vector of angular momentum. To describe the yawing motion, we must follow the motion of $\underline{L}$ (as affected by the yaw and other factors), superposing upon it the rotation of the quasi-nutation (about the instantaneous, moving, position of $\underline{L}$) at the steady angular velocity Ω. (This is equivalent to a numerical solution of an alternate form of the differential equations of motion). This decomposition, however, should not be confused with the customary decomposition of yaw into nutation and precession (which is possible only in the linear theory). The quasi-nutation and quasi-precession are not normal modes of motion, are not independent of each other, and are not constant in magnitude and rate (as the nutation and precession are). In the epicyclic motion the quasi-nutation (if considered separately) and the quasi-precession describe not the simple spirals, but epicycles (which, however, are closer to circles that the epicycle of $\underline{A}_1$ is). An additional shortcoming of our presentation is that for slowly-spinning fin-stabilized projectiles the magnitude of both the quasi-nutation and of the quasi-precession is near 90°, and the situation is somewhat less vivid, since the (small) yaw is effectively a small difference between two large quantities.

While we spoke of these quantities (nutation, precession, quasi-nutation and quasi-precession), rather loosely, as "angles", they are not, of course, simple angles (scalars); rather, each one of them should

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* Unless we go into quite advanced mathematical theory. Recently L. H. Thomas (Ballistic Research Laboratory Report No. 839, "The Theory of Spinning Shell", 1952) has introduced - by the method of variation of constants - the concepts of nutation and precession into the general, non-linear, theory.

properly be specified by two numbers (magnitude and orientation). The situation is again particularly simple in the linear theory, where these quantities are complex numbers, or "planar vectors", and are commutative (the order in which they are added is immaterial). Generally, these concepts are not very elementary (they are, strictly, quaternions or dyadics), but for our present purposes they are rather obvious concepts of spherical trigonometry.

METHODS OF PRESENTATION

The interpretation of the yawing motion in terms of the rolling of one cone upon another, cannot, unfortunately, be extended to the general, non-circular, yawing motion. We therefore have to resort to the above-mentioned abstraction, viz., to consider the motion of the shell axis $\underline{A}_1$ instead of the motion of the rigid body as a whole. The latter motion can always be visualized subsequently, either by superposing the axial spin ω_1 upon the motion of $\underline{A}_1$, or by constructing the vector $\underline{\omega}$ from the instantaneous positions of $\underline{A}_1$ and $\underline{L}$, and (1).

Let us imagine a sphere of unit radius, centered on the center of gravity of the shell; the sphere moves with the shell, but does not rotate ("remains parallel to itself"). Let us view the interior surface of this sphere from its center, and let us attempt to sketch the relationship of the vectors based on that center, as it would then appear. The directions in space would now map as points, and the angles as certain "straight" lines (the arcs of great circles). On such a sketch we will be losing the distinction between the vectors of certain definite length, and the unit vectors (which terminate on the surface of the sphere, and which we distinguish by the subscript 1); thus, the vector of the velocity $\underline{u}$ of the shell is coincident with the vector $\underline{T}_1$, etc. The exact manner in which the spherical surface is mapped on our sketches is immaterial at this stage: the surface of our sketches can be identified with the spherical surface only when the angles involved are small (we are then back in the linear theory), and whenever the angles are not small, we must revert to the three-dimensional representation. Some of our angles are inherently large: e.g., the vector $\underline{M}$ is at 90° to both $\underline{T}_1$ and $\underline{A}_1$, vector η_1 is at 90° to $\underline{A}_1$, etc.; it would be awkward to represent such vectors as points on our plane sketches, but we can readily visualize them as arrows projected on that part of the sphere at which we are looking. Since such a projection can be made, so to say, in any desired direction, the vectors so projected need not be confined to any particular line on our sketches, but must have only a particular direction.*

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* In exact terminology, the polar vectors (such as velocity and force) become axial vectors (such as angular velocity and torque) on our sketches. Obviously, we shall base torques on $\underline{L}$, forces on $\underline{T}_1$, etc.

Let us look forward as the shell flies, i.e., let us center our field of vision on the point (vector) T_1; the vector $\underline{T}_1$ does not have to be horizontal (though it is easiest to imagine it so). If the trajectory is a straight line (usually not a bad assumption), the point T_1 is fixed in the surface of our sphere, and is a very convenient point of reference. However, if the trajectory curves or swerves, the point T_1 moves. The velocity of T_1 is easy enough to compute. The tip of the vector of (linear) momentum of the shell, $mu\underline{T}_1$ (where m is the mass, and u the velocity of the shell), moves with the velocity equal to the applied total (gravitational and aerodynamic) force. Hence, that component of the applied force which is perpendicular to $\underline{T}_1$, divided by mu, is the (vectorial) velocity of T_1 on the surface of our sphere. We should also bear in mind that the total force may have a component along $\underline{T}_1$ (e.g., drag) which can cause additional effects.

Similarly, the velocity of the point L (the point of intersection of vector $\underline{L}$ with the surface of the sphere) is the component of the total applied torque normal to $\underline{L}$, divided by L (the magnitude of $\underline{L}$); again, there may exist other effects due to a component of applied torque along $\underline{L}$. It is interesting to note that the vector of applied torque must be decomposed, therefore, along and across $\underline{L}$ - while it is more natural (and in fact, is customary) to decompose it along and across $\underline{A}_1$.

While the vector of cross-spin η can be projected as an arrow tangent to the surface of the sphere at the point A_1, it is simpler to use, instead, an arrow representing the velocity of the point A_1 in the surface of the sphere. In magnitude this is η, and in direction this is, of course, 90° counterclockwise (on our sketches) from the projection of the vector η; and 90° clockwise from LA_1.

In order to systematize somewhat the forbidding possible multiplicity of the arrangements of the points A_1, T_1 and L on our sketches, let us also adopt this convention: let us, as it were, tip out head sidewise in the plane of yaw. To wit, we shall draw the point A_1 always directly above the point T_1; this will not mean that the plane of yaw is vertical - it is just easier to imagine it so. A naive attempt to depict the reader in the process of visualizing an instantaneous motion of a shell is given in Figure 4.

Let us adopt one more convention. With the single point A_1 on our sketches representing all possible yaws, we must still distinguish various possible directions of the velocity of A_1: i.e., we must distinguish the increasing and the decreasing yaws (or, an instantaneously circular yawing motion); as well as right-handed, left-handed and planar yawing motions.

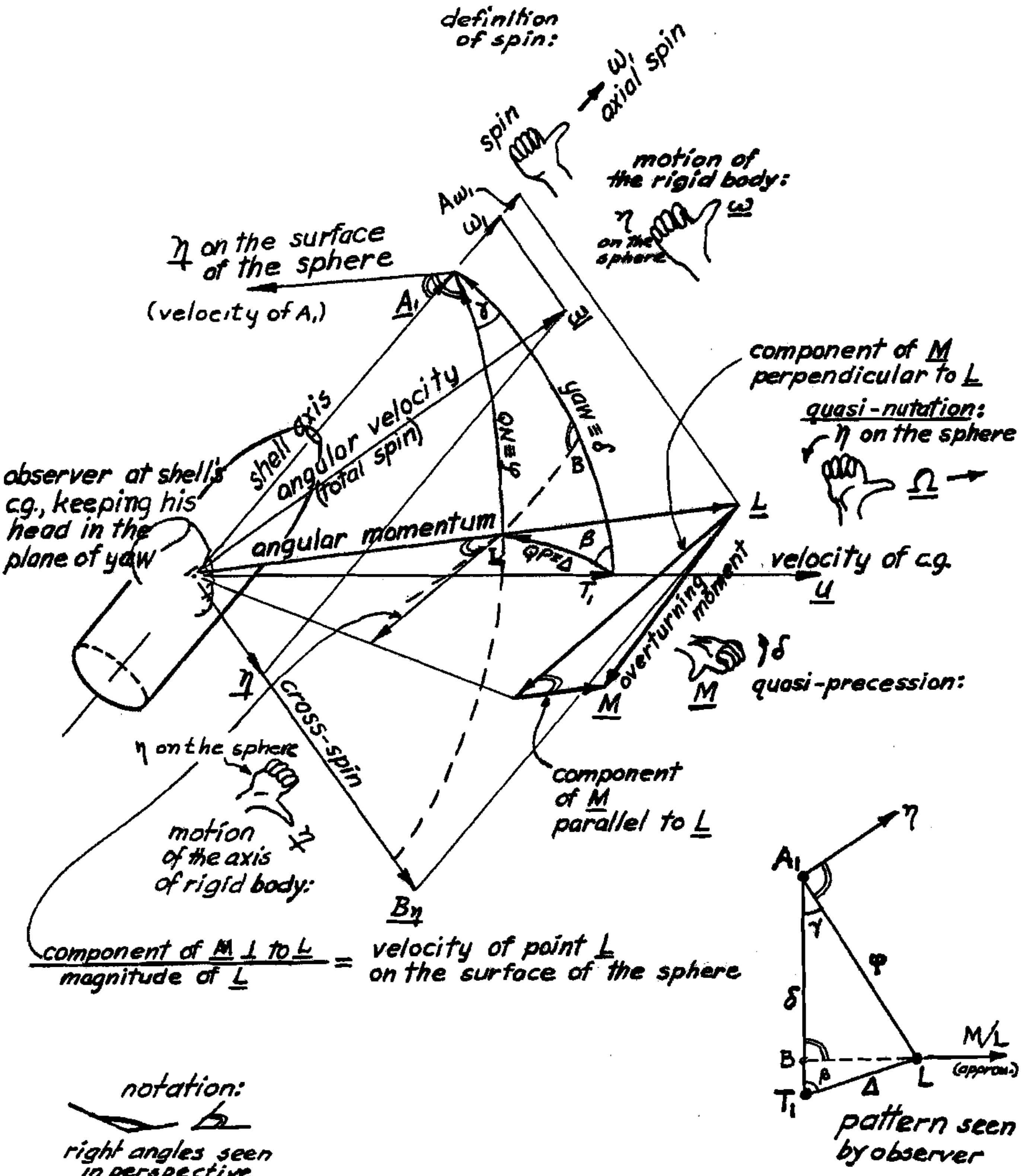


FIG. 4. Methods of Visualization

On our sketches, to specify the direction qualitatively it is easiest to refer to the points of the compass. Let us then repeat each sketch four (or eight) times, with different possible directions (N, NW, W, SW, S, etc.) of the velocity of A_1; and let us place each component sketch in the corresponding direction from the center of the sketch as a whole. Accordingly, our sketches will be arranged in the following scheme, whereby the instantaneous yawing motion is respectively:

increasing left-handed	increasing instantaneously planar	increasing right-handed
instantaneously circular left-handed		instantaneously circular right-handed
decreasing left-handed	decreasing instantaneously planar	decreasing right-handed

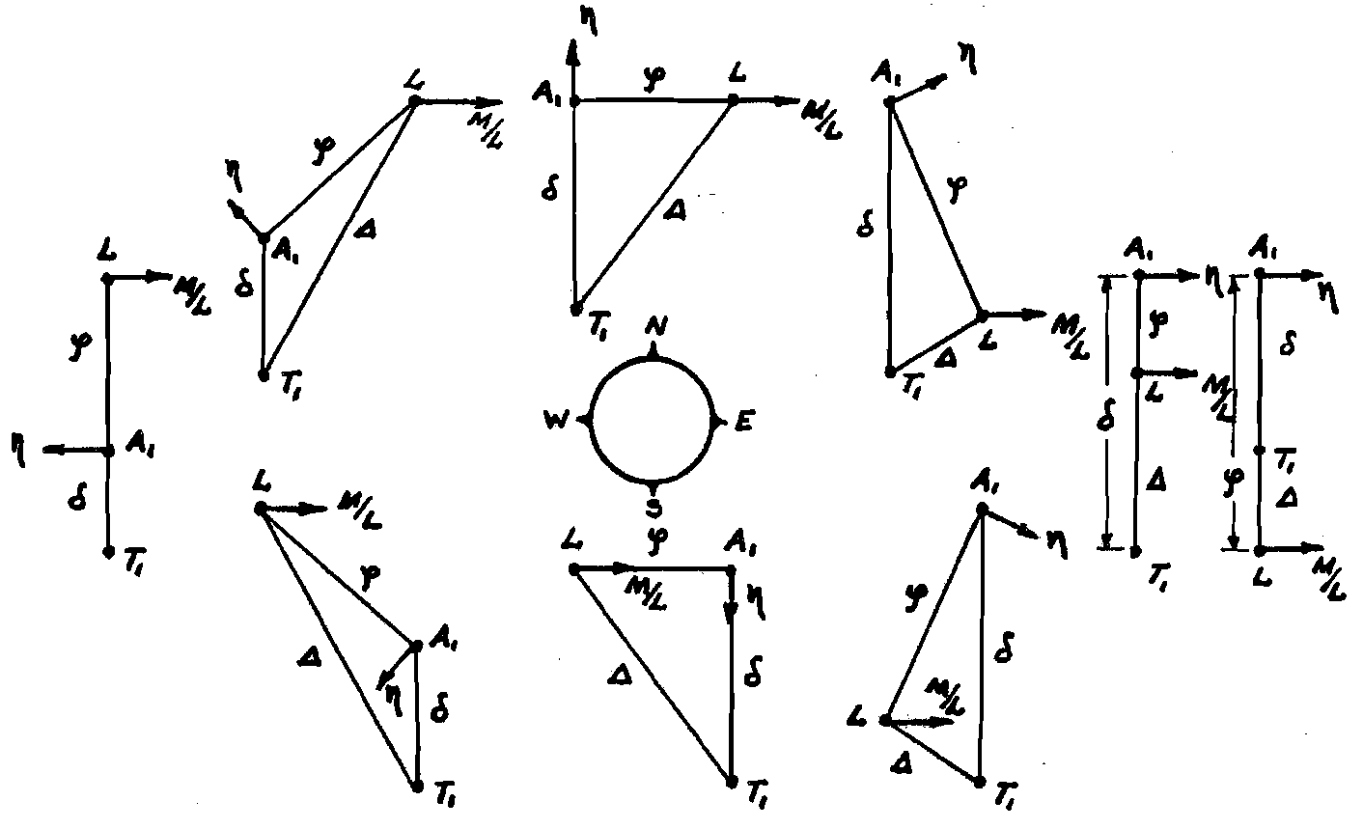
In order to become thoroughly familiar with such a method of presentation, let us now consider an example: let us investigate qualitatively the mechanism of the familiar looped epicycle of a spin-stabilized shell with right-handed spin.

MECHANISM OF AN EPICYCLE

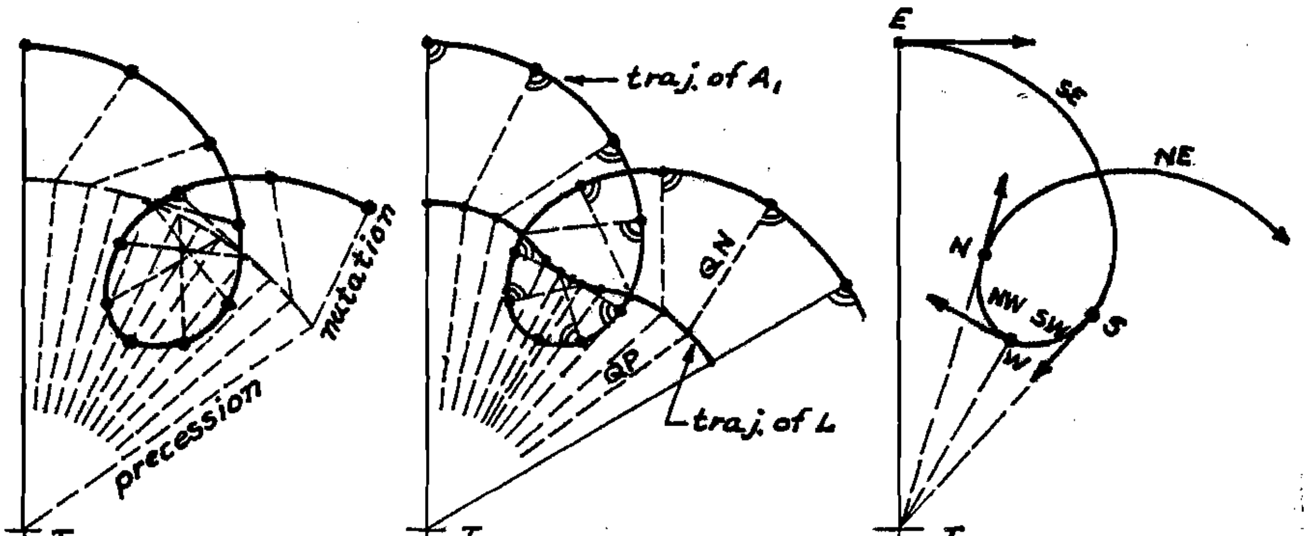
Let us refer to Figure 5a, and in particular, commence at the instant when the instantaneous yawing motion is circular and left-handed. This corresponds to the W sketch, and - as it will be presently seen on Figure 5 b - to the minimum of yaw.

Point L now must be above A_1. The overturning torque M moves L to the right, and the situation, obviously, passes into that described on the NW sketch.

Now the situation has very pronounced features of what we shall call "local instability". The yaw (T_1A_1) is growing, and particularly, in growing at an increasing rate: for, not only is L to the right of A_1, but also: the velocity of A_1 is becoming, on our sketch, more nearly vertical; both the quasi-precession and - particularly - quasi-nutation are growing, as though stretched, to some extent, by the torque M, so that with the constant angular velocity Ω of the quasi-nutation the linear velocity of A_1 is increasing; and finally, as the yaw is growing, the overturning torque M is growing, and also is turning to the left, and is thereby speeding up the velocity of L (it might be also noted that presently M begins to stretch L, increasing thereby the angular velocity Ω cf. equation (2). If such a situation were to continue long, the yaw would be quickly built up to very high values (which it



(a) sketches representing possible regimes of yawing motion



(b′) precession & nutation (b″) quasi-precession & quasi-nutation (b‴) regimes of fig 5a

(b) an example : usual epicycle

Fig 5: Method of Presentation

indeed would if the spin were too low, so that $\Omega = L/B$ is small and the velocity M/L of L is large). In practice, however, the situation (because of sufficiently high spin) rapidly passes into that described on the N sketch, and thereafter into that of the NE sketch.

Now certain inkling of stability becomes felt. If the spin is sufficiently great, the angular velocity Ω of quasi-nutation is also great, and that of quasi-precession (essentially the velocity M/L of L divided by the magnitude of the quasi-precession), small. The plane of yaw then begins to catch up with the quasi-precession. However, since the yaw is still growing, the velocity M/L is still increasing. It is still possible, therefore, that the yaw might not catch up with the quasi-precession (viz., A_1 might not catch up with L). It makes only a theoretical difference if it does catch up eventually, but only so late that the yaw builds up to a large value. If it fails to so catch up (and in practice, if it catches up too late), the shell is unstable. Note that throughout this NE regime the magnitudes of the quasi-nutation and of the quasi-precession continue to grow.

Generally, however, A_1 catches up with L, and the situation passes into that of the E sketch, i.e., an instantaneous right-handed circular yawing. The quasi-nutation and the quasi-precession now cease their growth. It is now that the stability of the shell is finally and truly manifested; in fact, it now becomes of interest to formulate the quantitative relations underlying the stability (this we shall presently proceed to do).

Thereafter, of course, the situation passes into that represented by the SE sketch; the yaw, the quasi-nutation and the quasi-precession begin shrinking in magnitude. This continues through the regime described by the S sketch and into that of the SW sketch; after which L overtakes A_1 again, and the situation passes into the W sketch again. Throughout the SE, S and SW regime the kinetic energy due to the cross-spin η is being converted into the potential energy of yaw, as the shell is brought, against the overturning torque, to face more nearly into the wind. Thereafter the potential energy slips back (in the NW, N and NE regimes) into the kinetic energy of cross-spin.

At this stage we do not care (as yet) to assert that the shell returns to its original angles of yaw and that the motion has a periodic character - even though this is indeed the case once our assumptions are properly crystallized. In reality there exist a number of complicating factors. Thus, the successive loops might become smaller, and eventually become cusps; the epicycle then becomes a "rosette", then a wavy circle, and finally passes into a precession similar to that described above. Or, the loops might grow; the epicycle would then pass through T_1; thereafter the loops would encircle T_1, become more nearly like coincident circles, and the epicycle passes into nutation. Again, the epicycle might shrink (or grow) retaining its shape. What matters for our present

purposes is that this, roughly described, mechanism of the motion survives in any case; and that any changes in the motion of A_1 can be accounted by the changes in the motion of L and T_1.

STABILITY

Concept

A theory of yawing motion of a shell is essentially the theory of stability of a shell. While much has been written on the concept of stability, the situation still seems unsatisfactory in spots. The word conveys an idea that is necessary and useful, but vague and having many ramifications. The tradition of mathematical approach demands that the word be associated with a definite quantitative criterion: for instance, the shell is considered (in the linear theory) stable (in a certain sense) if the so-called "stability factor" is greater than 1 (and then again, it can be less than 0 - and also, slightly less than 1 for spin-stabilized rockets). The trouble is, there are so many ramifications. We have, for instance, the static stability (fin-stabilization), gyroscopic stability (spin-stabilization), dynamic stability (the requirement that both precession and nutation are damping out - distinctly a concept of linear theory); many ingenious combinations of these requirements are in use. The subject of the trailing of the shell, obviously, is an aspect of stability. Accustomed to thinking in terms of the linear theory, we are wont to take it for granted that we can separate the question of stability from the question of the magnitude of yaw (that is in effect dismissed as a question of the constants of integration); but it is awkward to speak of the stability of a shell which is performing a steady nutation of 85° amplitude, and it is unfair to speak of the instability of a shell which in its whole trajectory changes its yaw from 1° to 2°. In the non-linear theory the concept of stability becomes dependent upon (i.e., inseparable from) the magnitude of yaw and other initial conditions. In the more advanced theories there exists a need for still other stability criteria, that have not as yet been formulated. Thus, in fact, the complexity of the problem defeats the good intention of making an intuitive idea concrete.

We feel, nevertheless, that the intuitive idea of stability of a shell is prior to, and independent of, all quantitative criteria. We therefore propose that the essence of the concept of stability is a prediction: an assurance that the yaw will not reach large values.

Naturally, a prediction can be only as good as the information on which it is based. If - like a Maxwell demon - we knew literally all there is to know about the shell at a given instant we would surely be able to predict exactly the behaviour of the shell for the remainder of its flight. The more advanced theories of exterior ballistics can do no more than attempt to approach this situation. The linear theory obviates the difficulty (of our ignorance) by the simple expedient of resorting to clearly specified assumptions; it is precisely this apparent ambition of the linear theory which gives to the student of

ballistics the false hope that a complete prediction is possible (and which leads to the natural - and at this time apparently well-entrenched - demand that any theory give a complete prediction). The fact is, the accuracy, and the surety, of a prediction depends upon the range (say, the interval of time) to which this prediction is to apply. In most practical cases, probably, the linear theory can give an excellent prediction for, say, several periods of yaw (e.g., a linear-theory epicycle can be fitted to well-nigh every stretch of several periods of yaw). For a time interval of, say, one eighth of the period of quasi-nutation the reader can make a fair prediction simply by visualizing one of our sketches. But in the very general case (say, a highly complicated force system) a good prediction can be made only for a very short time interval.

What, then, is the meaning of stability? Obviously, it must be viewed in relation to the length of time to which our prediction will apply. We may speak of "local" (instantaneous) stability; of, say, "short-range" stability (the assurance that, if the yaw is increasing, the increase will come to a stop and will be followed by a decrease); and finally, of various degrees of "long-range" stability (an assurance that the subsequent maxima of yaw - second, third, perhaps hundredth, will not exceed a specified number.

In the regimes of SE, S and SW sketches the local stability is inherent in the initial conditions: the yaw is decreasing. In the regime of NW sketch the stability, even if it does exist in some sense, is not manifest. Even if the NW regime passes into the NE regime, the stability is not certain; stability can be implied only by the assurance that the motion will indeed reach the situation described by the E sketch, when the point A_1 rolls, so to say, over T_1. Let us now inspect, quantitatively, the situation at such an instant.

Formulation

Let, on the E sketch, the magnitude of yaw (A_1T_1) be δ, the magnitude of quasi-nutation (LA_1) be ϕ, let M be positive (spin-stabilized shell) and let T_1 be stationary (rectilinear flight). The velocity η of A_1 is $\Omega \sin\phi$, or $(L/B)\sin\phi$ (cf. equation (2)); the angular velocity of A_1 about T_1 is $(L/B)\sin\phi/\sin\delta$. The velocity of L is* M/L; the angular velocity of L about T_1 is $(M/L)/\sin(\delta - \phi)$. The stability requirement is

$$(L/B)\sin\phi/\sin\delta \geqslant (M/L)/\sin(\delta - \phi),$$

and (since in this example M is positive, and no angle exceeds π), can be written as

$$\frac{(L/B)\sin\phi/\sin\delta}{(M/L)/\sin(\delta-\phi)} = \frac{L^2\sin\phi\ \sin(\delta-\phi)}{B\ M(\delta)\ \sin\delta} \geqslant 1, \qquad (3)$$

where M is written in the form which reminds us that it is the function of of δ. This is the stability factor relation in a generalized form (although

*Generally, it is $M\cos(L,B)/L$ (cf. Fig. 4); but for the E sketch the angle (L,B) is zero.

the left-hand side of this inequality is not necessarily the stability factor, as shall be explained presently).

Linear Skeleton

For the sake of both simplicity and check, let us revert briefly to the assumptions of the linear theory. Then $\underline{L} \cong A\omega_1$ (because of the smallness of ϕ - cf. Fig. 1), $M = \mu\delta$ (μ a constant), and sines are equal to their angles; (3) becomes

$$\frac{A^2\omega_1^2}{B\mu}(\phi/\delta)(1 - \phi/\delta) \geqslant 1 \qquad (3')$$

The product $(\phi/\delta)(1 - \phi/\delta)$ has a maximum value of $1/4$, occurring at $\phi = \delta/2$; even with this ϕ, there remains to be satisfied the inequality

$$s \equiv \frac{A^2\omega_1^2}{4B\mu} \geqslant 1, \qquad (3'')$$

which is the definition of the stability factor s. The physical significance of the stability factor is thus particularly simple: for a given combination of A, B, ω_1 and μ, the stability factor is the maximum possible ratio of the angular velocity of the plane of yaw to the angular velocity of quasi-precession. This ratio occurs only at very particular circumstances; at the maximum of yaw, and when the vector of angular momentum bisects the angle of yaw. It is at these circumstances that the stability is most clearly manifested. It is interesting to note that the epicycle of the linear theory in these circumstances passes through the origin; i.e., this is the familiar case resulting from the initial conditions $\delta = 0$, $d\delta/dt \neq 0$.

Given $s > 1$, (3') can be satisfied not only at $\phi = \delta/2$, but also in a certain neighboring range of ϕ. At the extremes of this range (3') is barely satisfied, i.e., becomes the equality

$$4s(\phi/\delta)(1 - \phi/\delta) = 1,$$

which can be rewritten as

$$(\phi/\delta)^2 - (\phi/\delta) + 1/4s = 0, \qquad (3A)$$

and has the two solutions,

$$(\phi/\delta)_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{1 - 1/s}\right) = (1 \pm \sigma)/2, \text{ say}, \qquad (3B)$$

These two extremes are, of course, nutation and precession. In nutation L is closer to T_1, or ϕ/δ is more than $1/2$; A_1 moves fast because of the large ϕ, and L just barely keeps up with A_1. In precession L is closer to A_1, and, because of the small ϕ, A_1 moves

only slowly, and just barely keeps up with L. The instantaneous motion of the body in both cases is rotation about $\underline{\omega}$ at the same angular velocity ω; or, the motion of A_1 in both cases is rotation about $\underline{L}$ at the same angular velocity Ω_L. But the reader should remember that these angular velocities describe the rotation (of the body, or of the "vector" of quasi-nutation) about the instantaneous (i.e., stationary, as it were) position of ω and L, and that the points ω and L are moving. The situation is simple enough with quasi-precession (L moves with velocity M/L), or with the rotation of the body (it is rolling), but with regard to the quasi-nutation the situation is not quite the same sort of thing for which one is accustomed. The concept of the angular velocity of the quasi-nutation could have been introduced as the angle between two subsequent positions of quasi-nutation divided by the time increment;* this was not done because such an angular velocity would not be the same as the constant Ω_L, and would not be as convenient; different such velocities can be produced by the same Ω_L.

The variation of the shape of epicycle in the range $(1 - \delta)/2 < \phi/\delta < (1 + \delta)/2$ is very interesting. If we decrease ϕ/δ below 1/2, the loops begin to miss the center, become smaller, and epicycle passes through a rosette to wavy circle and into precession; if we increase ϕ/δ over 1/2, the loops start going around the origin, increase, and the epicycle passes into nutation. In either case, the variation of the magnitude of yaw diminishes, until it finally disappears in the corresponding circular motion. Thus the rapid variation of the magnitude of yaw (as in the epicycle passing through the origin, with $\phi/\delta = 1/2$) is a token, or a manifestation, of stability.

Outside of this range of ϕ/δ the inequality (3') is not satisfied, and the stability, even if it does exist in some sense, is not manifest in the regime of the E sketch. This sketch then represents a minimum, rather than a maximum, of yaw. With the smaller ϕ/δ, the epicycle will be a wavy circle; with the larger ϕ/δ, the resultant epicycle will loop over the origin.

A very important borderline case is that of s = 1. Precession and nutation then merge, and only a bare manifestation, or a minimum, of stability (viz., circular motion) becomes possible; and that, only at

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* In fact, in the linear theory, when nutation is considered a vectorial component of yaw, this is exactly how the angular velocity of nutation is defined; it is useful there, for such angular velocity then happens to be constant. Presently, when we shall relax the assumption of the stationary T_1, we shall meet the same difficulty with the quasi-precession.

the most advantageous condition, $\phi/\delta = 1/2$*. Note that such motion is still possible at all (small) angles of yaw.

More General Cases

With this understanding of the skeleton of the "short-range" (in this case, gyroscopic) stability we may relax our assumptions of linearity, and return briefly to the more general problem, i.e., to (3). Now we have more exactly (subject, of course, to other assumptions here implied) $L = A\omega_1/\cos\phi$ (cf. Fig. 1), and $M = \mu\sin\delta$ (where μ so defined is not necessarily a constant, but is certainly more nearly so than M). Then (3) becomes

$$\frac{A^2\omega_1^2}{B\mu(\delta)} \frac{\sin\phi \sin(\delta-\phi)}{\cos^2\phi \sin^2\delta} \geqslant 1, \qquad (4)$$

which we shall write as $sf(\phi,\delta)\geqslant 1$, with $s \equiv A^2\omega_1^2/4B\mu(\delta)$ and $f \equiv 4\sin\phi\sin(\delta-\phi)/\cos^2\phi\sin^2\delta$. While (4) is still similar to its linear analog (3'), the important difference is that with the larger angles the trigonometric part of (4), or $f/4$, can have values greater than $1/4$, and the stability requirement is thereby made less strigent.

The very interesting function $f(\phi,\delta)$ is given, as a contour plot in the (ϕ,δ) plane for $0<\phi<\delta<\pi/2$, on Figure 6. The most important region of this surface is the region of small ϕ and δ, where the two contour lines of $f = 1$ merge; this is the domain of the linear theory. In this region the function has a practically horizontal crest, or watershed; this crest extends, as a razor wedge, right to the point $\phi = \delta = 0$, which represents a steady nose-on flight. However, on the boundary lines radiating from the origin the function is zero, and the manifestation of stability is impossible. In particular, the abscissa ($\phi/\delta = 0$) represents the situation when the axis of a spinning shell is momentarily stationary (as in the cusp of a rosette), and L is sweeping across the plane of yaw. The 45° line, $\phi/\delta = 1$, represents the situation when L sweeps across the trajectory, and therefore rotates at an infinite angular velocity, as it were, about the trajectory. In the linear region the crest lies along the line $\phi/\delta = 1/2$; on the extension of this line (which, for moderate yaws, is not far from the true crest) the function has the

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* The mathematical linear theory shows that when two independent solutions merge into a single one, as here, there arises, as the second possible solution, the "secular" solution; in this case, it is an expanding Archimedean spiral. When s becomes less than 1, both the circular and the Archimedean solutions pass into logarithmic spirals (respectively, shrinking and expanding). The blend between various types of solutions can be inspected as a matter of initial conditions (constants of integration). For instance, with arbitrary initial conditions and with s barely over 1, the NE regime will be very long, and the maximum angle of yaw very large; with s of 1 or less the NE regime will continue until the yaw reaches values so great that the linear theory no longer applies. We have avoided such difficulties by the simple device of considering only the initial conditions of the E sketch.

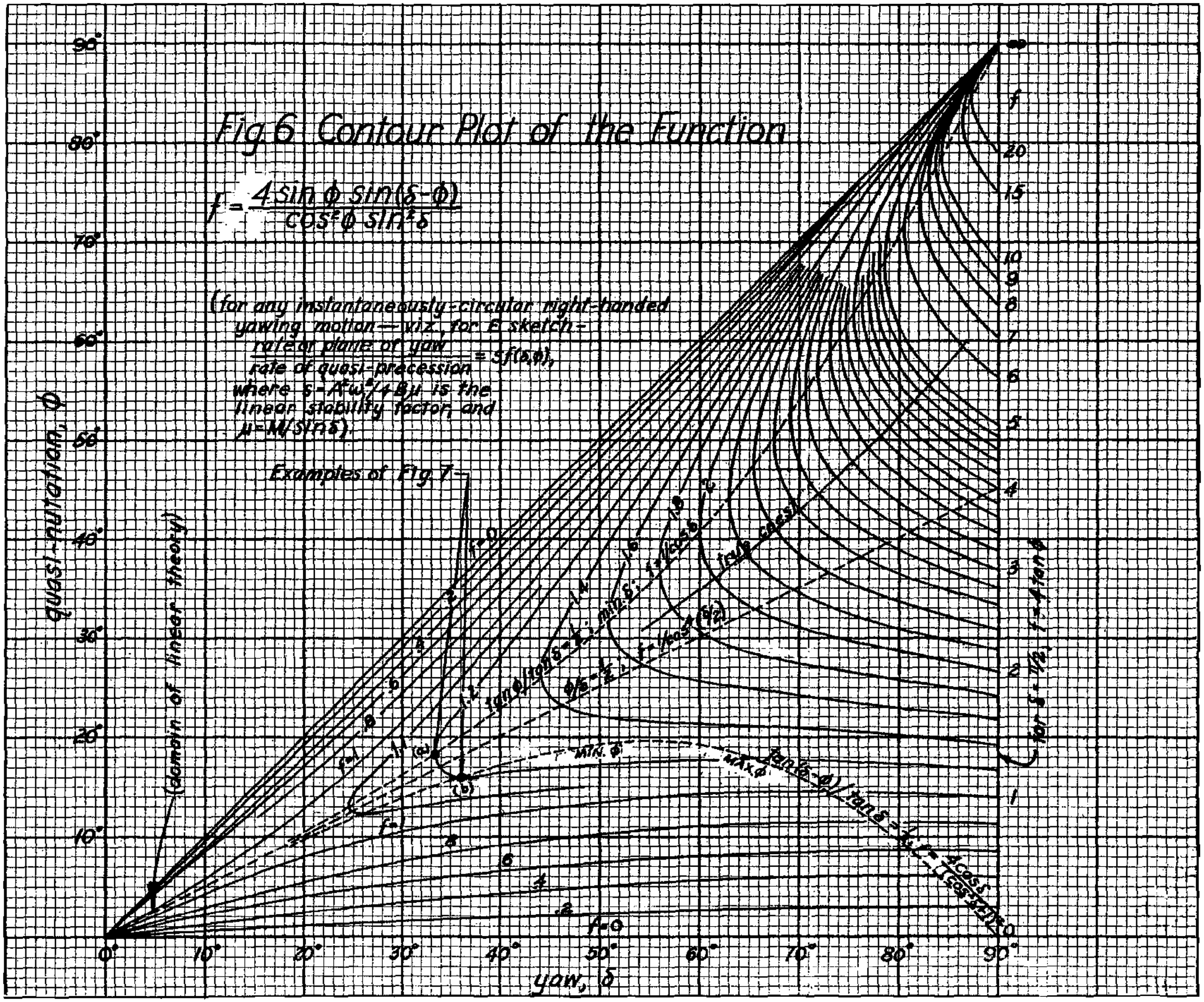

Fig. 6 Contour Plot of the Function

$$f = \frac{4 \sin\phi \sin(\delta-\phi)}{\cos^2\phi \sin^2\delta}$$

value $1/\cos^4(\delta/2)$, which (for these yaws) is only slightly smaller than $1/\cos\delta$.

The point $\phi = \delta = \pi/2$ represents a pure cartwheeling of the shell, i.e., cartwheeling without any axial spin. At this point the function f may have any value from zero (since it is on the line $\phi/\delta = 1$) to infinity (since it is also on the line $\delta = \pi/2$, which represents the cartwheeling of the axis of the shell, and on this line f becomes simply $4\tan\phi$). Indeed, at such conditions manifestation of stability rather loses its meaning.

It will be seen that the true crest of our surface approaches this "infinite peak" (the singularity) in a rather devious way, first going outside of the limits of the triangle $0 < \phi < \delta < \pi/2$. It is important to note, however, that this "peak" can be reached without going outside of this triangle; i.e., all values of f are represented in this triangle. This means that for any shell which is unstable in the linear-theory sense there may exist combinations of ϕ and δ such that this shell will still manifest stability.

The stability condition can then be visualized as follows. For a given s (which we may now term "linear stability factor") we may imagine on Figure 6 a point with map coordinates ϕ and δ and at the height $1/s$. If this point is above the f-surface, the stability condition (4) can not be satisfied (the regime of the E sketch, even if does exist instantaneously, can not continue, and will pass into the regime of the NE sketch). If this point is in the f-surface (on the contour line $f = 1/s$), the circular yawing will continue (the stability is barely manifested). If this point is under the f-surface (inside the loop of the contour line $f = 1/s$), the stability is clearly manifested (the regime of the E sketch will pass into that of the SE sketch, and the yaw will presently start decreasing).

Naturally, the possible circular yawing regimes might be divided (say, by the crest of the function f) into those of mutation and precession; the rates of mutation and precession can thus be computed, either as the rate of the plane of yaw $(L/B)\sin\phi/\sin\delta$, or as the rate of the quasi-precession $(M/L)/\sin(\delta - \phi)$. We may thereby relax now the requirement (mentioned above as a matter of convenience of the presentation) that M be small. These rates, however, are no longer independent of the initial conditions (the magnitude and the angular rate of yaw). Moreover, these two regimes are no longer separated, on Figure 6, by the regime of the circular yawing motion of a barely stable projectile (as they are in the linear theory); rather, mutation and precession can now be blended across the crest of the f-surface. The multiplicity of possible solutions in the region of this blend corresponds to the fact that in the linear theory a barely stable shell may yaw circularly at all (small) angles of yaw.

It may be noted on Figure 6 that the true crest of the f-surface does not represent exactly the minimum values of δ at which a given linearity-unstable shell may manifest its stability. The interesting question of what these values are has a particularly simple answer: the locus of the minima of δ can be found by finding the maxima of f

in the planes of δ = constant. The ϕ-dependent part of (4) can be written as

$$\sin\phi(\sin\delta\cos\phi - \cos\delta\sin\phi)/\cos^2\phi = \cos\delta \tan\phi \ (\tan\delta - \tan\phi),$$

which is of the form $ax(b - x)$, and has a maximum at $x = b/2$, i.e., at $\tan\phi = (\tan\delta)/2$. Thus, the vector of angular momentum then bisects the tangent of the angle of yaw erected on the axis of the shell (Figure 7a), rather than the angle, as in linear theory. This locus merges, of course, with the line $\phi/\delta = 1/2$. At these conditions the value of f is simply $1/\cos\delta$. Therefore, the generalization of the stability condition (3") of the linear theory is

$$s/\cos\delta \geqslant 1, \qquad (4')$$

with the proviso that at the critical δ stability may be manifested only if the yawing is instantaneously circular and ϕ has a particular value. This proviso corresponds to the fact that in the linear theory a barely stable shell may manifest stability only at particular initial conditions, viz., at $\phi/\delta = 1/2$.

In fact, in advanced texts on ballistics the stability factor is defined with $\cos\delta$ in the denominator, i.e., as the left-hand side of (4') rather than by (3"); we might then speak of the "non-linear" stability factor. This is quite a different thing from our linear stability factor, and no longer can be viewed as a definite property of the shell, independent of the initial conditions (as the linear stability factor is usually considered); rather, it should be viewed as the maximum possible ratio of the angular velocities of yaw and quasi-precession.

The requirement of a sufficiently high yaw (δ) may be interpreted as a requirement* of a sufficiently large quasi-nutation (ϕ) and a sufficiently large quasi-precession ($\delta - \phi$). It might complete a certain symmetry of our arguments if we inquire, briefly, if loci of such minima exist. In fact, the locus of minima of ϕ is quite pronounced on Figure 6; it exists from the linear domain to δ of some 56°. Its equation is $\tan(\delta - \phi) = (\tan\delta)/2$; i.e., for stability ϕ must be sufficiently large at least to bisect the tangent of the angle of yaw erected on the trajectory (Figure 7b). For larger δ this curve represents the maxima, instead of the minima, of ϕ on the contour lines of f. The locus of the minima of $\delta - \phi$ is barely noticeable in the region from the linear domain to δ of some 25°, and seems to be of no particular interest.

The usefulness of the stability factor lies in its relation to the nutational and precessional rates. In the linear theory these rates are - from (2), Figure 1 and (3B) - simply

$$\eta/\delta = \Omega\phi/\delta = (A\omega_1/B)(\phi/\delta) = (A\omega_1/2B)(1 \pm \sqrt{1 - 1/s}) \qquad (5)$$

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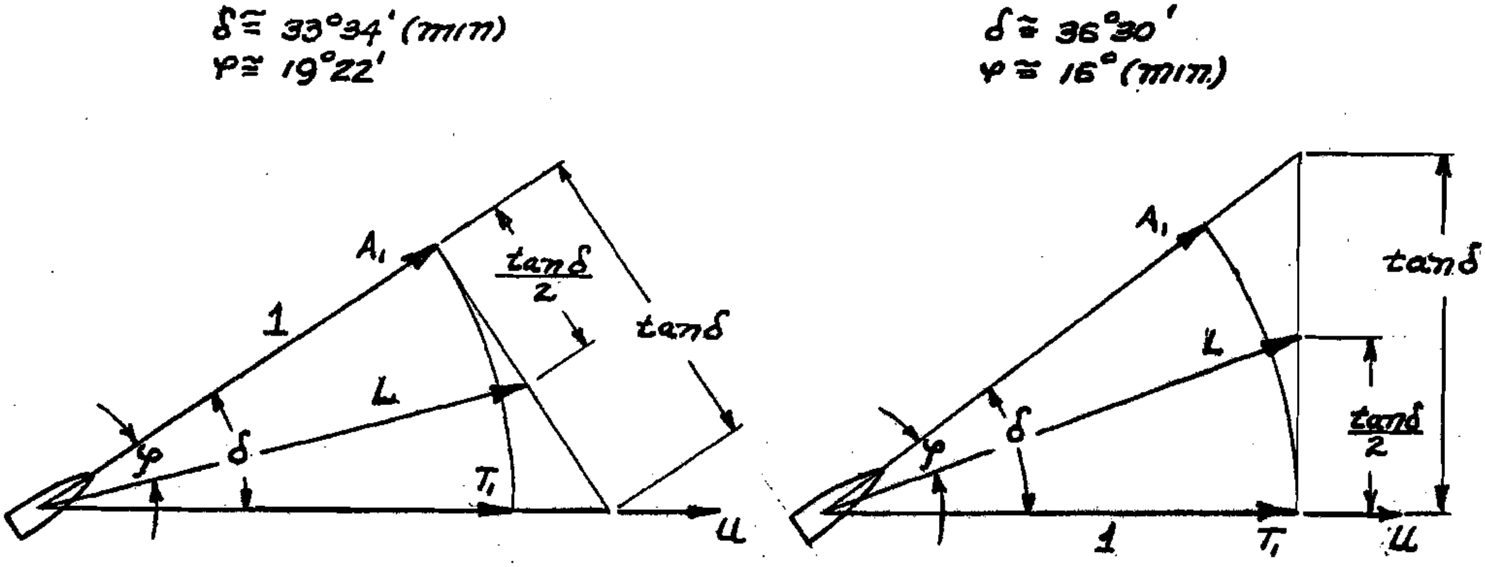
* For any $s < 1$.

$f = 1.2\ ;\ s = 5/6$



(a) at the minimum angle of yaw

(b) at the minimum quasi-nutation

Fig 7: Examples of Non-Linear, Barely-Stable Motions of Linearly-Unstable Shell

For any circular motion the non-linear analog of (3A) can be shown to be

$$\tan^2\phi \cot\delta - \tan\phi + \sin\delta/4s = 0, \qquad (4A)$$

so that the analog of (3B) is

$$\tan\phi/\tan\delta = (1 \pm \sqrt{1 - \cos\delta/s})/2, \qquad (4B)$$

and the circular rates are

$$\eta/\sin\delta = \Omega\sin\phi/\sin\delta = (A\omega_1/B)(\tan\phi/\sin\delta) = (A\omega_1/2B)(1 \pm \sqrt{1-\cos\delta/s})/\cos\delta^{*} \qquad (6)$$

<u>Example</u>

The foregoing discussion was mainly the recital of the more interesting conclusions from the theory of spinning top - for which case the assumption of a constant μ is particularly justified. While the analogy between the shell and the top is quite traditional, a comparison between the linear mathematical analog and the every-day experience with a top is met rather less frequently; such a comparison might be rather illuminative.

In the actual top the nutation damps out extremely rapidly (by processes of which more anon, in connection with Magnus torque), so that in practice it is observed seldom if ever. The precession also damps out quite rapidly; both the damping and the angular rates of precession are readily sensed when we see a hard-spinning top right itself and "go to sleep". Yet the reader will recollect that the top often fails to reach the sleeping position, and continues a rather steady "precession"; also as the spin of a sleeping top is dying out, there invariably develops the same, relatively slow, precession - which speeds up as the yaw increases, just before the top falls. These familiar "steady" motions, apparently, are not the precession in the customary ballistic sense (which is a result of disturbance, expressed by a solution of a homogeneous differential equation); rather, they are the motions of a linearly-unstable top, which is barely stable (has the non-linear stability factor $s/\cos\delta$ of 1) at the given angle of yaw. Any oscillations

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* The apparently peculiar behaviour of the last expression in (6) at $\delta = \pi/2$ can be readily accounted for by inspecting the legitimacy of our mathematical manipulations at this δ. The precessional rate (which appears indeterminate) then is simply $A\omega_1\tan\phi/B$, i.e., η. That the nutational rate is not infinite can be seen as follows. From (4B) or Figure 6 it is seen that nutation at $\delta = \pi/2$ is possible only if $\phi = \pi/2$, i.e., if the axial spin ω_1 is zero, and the apparently indeterminate (6) then amounts to η again, - with the additional proviso that if the motion is to be circular, as assumed, M must be zero. The trick difficulties of this kind are inspected best by examining the physical significance of the problem (e.g., Figures 2 and 3), rather than the legitimacy of algebra.

of this angle rapidly damp out, apparently by processes similar to the damping of the ballistic nutation and precession; in case the non-linear stability factor is less than 1, the top merely builds up its yaw till it is non-linearly barely stable at some larger yaw; the speed-up of the precession (in spite of the decrease of ω_1) just before the top falls is the consequence of the presence of $\cos\delta$ in the denominator of (6). Thus it is the usually-neglected border-line case of the linear theory, rather than the theoretical epicycle, which is the ancestor of the commonly-observed motions in the non-linear regime. It is extremely easy to confuse this motion of non-linear regime with the ballistic (linear) precession; i.e., to consider it, erroneously, a matter of initial conditions. A shell with an insufficient linear stability factor may not exactly "tumble"; it is still "stable" in a sense. It is just that its yaw is undesirably large.

PREDICTION OF THE YAWING MOTION

In the tradition of mathematical approach the stability condition stands somehow apart from the solution; although it is understood that it is also a kind of a short-cut summary of the character of the solution. We interpreted the stability condition as an attempt at the prediction (i.e., the solution) itself; and we found that a consideration of an instantaneous situation may, in a number of important special cases, yield a modicum of a long-range prediction. We avoided a quantitative discussion of the local stability (viz., of the question whether a prediction for a short length of time augurs well or ill for the decrease of yaw); and discussed quantitatively only the short-range stability, which is, as it were, the local stability at the instant of the first maximum of yaw. But to get to the first maximum of yaw from arbitrary initial conditions, as well as to get from the first maximum to the second, third, etc., is a different, and a more difficult, problem; in fact, this is the problem which has been solved analytically only in the most simple cases (such as the linear theory, or the case of the M-torque acting alone), and which requires, generally, modern high-speed computing machinery.

In the linear theory the general solution is obtained by the superposition of the simple nutation and precession; and the superposition arises naturally, as a property of solutions of a linear differential equation. As a matter of a mere exercise, or of an inquiry how far we can go without calculus, the following interpretation, or a derivation of the superposition might be of insterest.

For any yaw ζ (which in the linear theory is simply a complex number) there exist two possible rates, say $\dot{\zeta}_n$ and $\dot{\zeta}_p$, at which the motion would be the simple (circular) motion; for each one of these possible rates the given ζ must be decomposed into quasi-precession and quasi-nutation in a definite way - which, as we have seen by (3A), is independent of ζ. The actual instantaneous rate $\dot{\zeta}$ of ζ, generally,

is neither one of these. Now, two complex numbers, $\dot{C}_n$ and $\dot{C}_p$, can easily be found such that $C_n + C_p = 1$, and $\dot{\zeta} = C_n \dot{\zeta}_n + C_p \dot{\zeta}_p$. Obviously, $\dot{\zeta}$ may be decomposed into $C_n \dot{\zeta}$ and $C_p \dot{\zeta}$; a brief geometrical or algebraic consideration, detailed on Figure 8, will show that the quasi-mutation and quasi-precession may be decomposed similarly (e.g., on Figure 8, $QP = C_n QP_n + C_p QP_p$); the vectors QP_n and QP_p are designated on Figure 8 by L_n and L_p. Also, the projected vector of the torque M may be decomposed in the same way.

Now, it does not matter which component of torque acts on which component of angular momentum, as long as the total $\underline{M}$ acts upon the total $\underline{L}$. Furthermore (but this holds only in the linear theory), it does not matter how the quasi-mutation is decomposed. All of its components are rotating (about the corresponding stationary positions of L) at the same angular velocity Ω, and the velocity $\dot{\zeta}$ (or η) of A_1 is, in any case, the vectorial sum of the velocities produced by the components of quasi-mutation. We therefore can assign, arbitrarily, the following relations. Let the torque produced by the $C_n \zeta$ component of yaw affect the $C_n QP_n$ component of quasi-precession QP; and let the angular momentum component, represented by that component of quasi-precession, produce - by means of the corresponding component $C_n QN_n$ of quasi-mutation QN - the component $C_n \dot{\zeta}_n$ of $\dot{\zeta}$; with similar relations holding, of course, among the components with the p-subscript. We can then note that among the components with each subscript the motion is known, i.e., predictable, circular motion. Therefore the C's will remain constant, and the general solution is a superposition of precession and mutation.

While this intrepretation is particularly obvious in a highly simplified case (rectilinear flight, constant velocity, only $\underline{M}$ present), it is not vitiated if we pass to the general (linear) case. We need only to re-label the components $C_n \zeta$ and $C_p \zeta$ as $C_n \zeta_n(t)$ and $C_p \zeta_p(t)$, introduce the yaw of repose and the more complicated system of torques and forces (all of which, however, must be linear homogeneous functions of ζ and $\dot{\zeta}$), introduce the motion of T_1, etc., and re-trace the argument more slowly and more laboriously. In fact, the component motions $\zeta_n(t)$ and $\zeta_p(t)$ may be quite complicated: e.g., such parameters of motion as axial spin, velocity and stability factor may all change - provided they change in the same way for both ζ_n and ζ_p.*

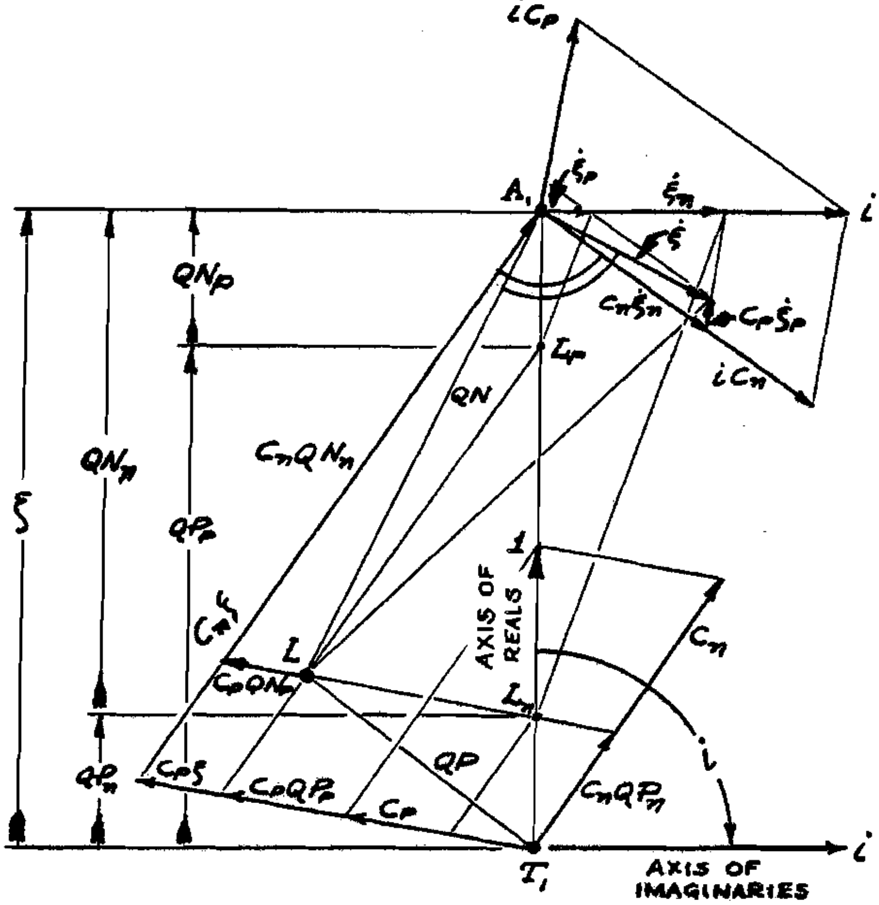
*The latter requirement rules out, in the linear theory, such phenomena as instability of spin (BRL 668), cases when the yaw-drag affects the trajectory, etc.; those are domain of non-linear theories. It is not quite idle to remark that the component motions need not even be precession and nutation ! They can, equally well, be any two linearly independent epicycles. Such procedures are indeed often resorted to, when various linear perturbations of flight data must be carried out.

$$C_n + C_p = 1;\ i = C_n i + C_p i = C_n(QP_n + QN_n) + C_p(QP_p + QN_p) =$$

$$= \underbrace{(C_nQP_n + C_pQP_p)}_{(QP)} + \underbrace{(C_nQN_n + C_pQN_p)}_{(QN)}$$

$$\dot{i} = C_n\dot{i}_n + C_p\dot{i}_p;\ \text{but}\ \Omega = \left|\frac{\dot{i}}{QN}\right| = \left|\frac{\dot{i}_n}{QN_n}\right| = \left|\frac{\dot{i}_p}{QN_p}\right|;\quad \therefore\ QN = C_nQN_n + C_pQN_p$$

$$\therefore\ QP = C_nQP_n + C_pQP_p$$



NOTE:

$QN_{n,p}/\xi = (1 \pm \sigma)/2$; $QN = (C_n\xi)(1+\sigma)/2 + (C_p\xi)(1-\sigma)/2$

$QP = (C_n\xi)(1-\sigma)/2 + (C_n\xi)(1+\sigma)/2$; $\dot{\xi} \perp QN$; $C_n\dot{\xi}_n \perp C_n$; $C_p\dot{\xi}_p \perp C_p$

By assuming C_n and C_p, instead of i_o and $\dot{i}_o$, this drawing is constructed more easily.

Fig 8 Mechanism of Superposition of the Linear Theory

Once the superposition is accepted, such facts as the absence of damping with M alone acting on the shell (and with M being a function of δ alone), as well as the complete prediction of motion, follow readily. In this connection it is well to note the distinction between the dynamic and the long-range stability. Consider a shell whose nutation damps rapidly, but whose precession grows, albeit very slowly. In the clean-cut tradition of the mathematical approach, such shell is automatically classified as dynamically unstable. Our approach seems (in that case) more tolerant, or more nearly attuned to the reality. If the initial yawing motion is mostly nutation, we would say this shell has a long-range stability for so-many periods of yaw, or that the instability would not be manifested until such-and-such range, or, again, that the stability depends upon the manner of launching. This is certainly no different from what would usually be done: it is just that our "long-range" stability corresponds to the final calculations of the linear theory - or, to proof-firing results - rather than to the abstraction of dynamic stability.

An attempt to systematize the resultant possible multiplicity of epicycles is given on Figure 17. It is amusing (and from our "non-mathematical" point of view, chastising) to observe that in practice the simplest way of drawing these curves has been to construct the appropriate differential equations, to compute the appropriate initial conditions, and to solve these equations numerically on an electronic differential analyzer (GEDA).*

Even allowing that the general non-linear case is not solvable analytically, it is of interest to inquire whether there might not be some simple non-linear cases, e.g., in the case of M acting alone, and being a function of δ alone. Unfortunately, the theory furnishes no simple and useful answers there.** Even if we are to prove the very relevant fact, that the yawing motion in such a case is not damped, it is difficult to keep up our pretence of avoiding the calculus.***

Let us now inspect the consequence of the various components of the force system acting upon the shell - qualitatively in the general case, and quantitatively in the linear case.

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* This work was done by Mr. E. E. Bomberger.

** As was mentioned before, the general solution has been worked out, but is cumbersome. In the case of spinning top (constant μ) the solution is in terms of elliptic integrals.

*** Yet we might attempt to outline the proof, very roughly, as follows. Since M is perpendicular to the shell axis, the axial spin cannot be affected; and since M is perpendicular to trajectory, the component of L along trajectory cannot be affected. It can be then shown that since M depends upon δ alone, there exists a potential energy of yaw, and therefore the system is what is known in mechanics as conservative; and such systems, generally, possess constant total energy and represent oscillations "in a potential well", with constant amplitudes. Such vague "reasoning" is not without its legitimate place in ballistics; in fact, it is more logical than, for instance, an insistence on a large stability margin of a mortar shell! This intuitive groping for "constants of motion" can be recognized as a primitive form of the Hamiltonian mechanics.

FORCE SYSTEMS OF BALLISTICS

In recent years the fact began to be recognized that, the force systems used in exterior ballistics were rather oversimplifications of reality.* Nevertheless, for practical purposes an idealization is, of course, indispensable; it is just that a realization of just what has been assumed, or taken for granted, must now be kept in mind more prominently.

The first restrictive assumption is that of Synge's aerodynamic hypothesis: that the force system (viz., total force and total torque) can be specified by specifying the vectorial velocity and angular velocity of the shell.** The next systematization is the Maple-Synge analysis of the consequence of the symmetry of the shell. This systematization is particularly useful in investigating the border-line between the linear and non-linear theories. It restricts somewhat the number of possible non-linearities which ought to be considered. The usefulness of this systematization in the general non-linear regime is, unfortunately, limited, for two reasons. It is difficult to design experiments so that all probably relevant Maple-Synge coefficients can be measured; and even if they were, the mathematical utilization of them would be extremely laborious.

There are two known ways in which a further necessary simplification of Maple-Synge theory may proceed. The most important one, of course, is the linearization *** of the Maple-Synge theory. This excludes a number of coefficients which are known to be relevant, and includes some whose relevance seems rather marginal. The most elegant feature of the linearization scheme (in addition to the fact that it dovetails with the superposition of the linear theory) is that it furnishes a convenient coordinate system for decomposition of the force system; forces and torques dependent upon δ are decomposed along and across the plane of yaw, and those dependent upon η are decomposed along and across the arrows of η in our sketches. This gives rise to the well-known Kelley-McShane matrices of forces and torques,

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* So that a number of extant ballistic texts now appear, in "retrospect", rather didactic.

** This assumption seems to be rather peculiar to ballistics. In the advanced aeronautics the attempts to rescind this assumption dates to the 1920's. Cf. BRL 882, "Dynamic Measurement on 81mm, M56 Shell in NBS Wind Tunnel", for reference and for brief discussion.

*** Chronologically, of course, the linearized (Nielsen-Synge) version of Maple-Synge theory preceeded the more complete theory.

$$
\begin{matrix}
DA & - & - & A \\
N & F & M & T \\
S & XF & H & XT,
\end{matrix} \tag{7}
$$

which will be discussed in detail presently. Its usefulness as a classification scheme survives, to some extent, even in the non-linear theory.

The second known way is the adoption of circular yawing motion (rather than small yaw) as the basic form of yawing motion. The most troublesome Maple-Synge coefficients are then excluded, and experiments can be readily designed by which all terms of the Kelley-McShane matrix can be measured - whereby this matrix can be considered non-linear, and merely an aid in classification. The mathematical utilization, unfortunately, remains difficult.

What is probably wanted first is an application of the Maple-Synge (or some other) theory of aerodynamic forces acting on a given surface of the shell (and independent of the mass distribution within the shell) to a class of "possible", or at least "probable", yawing motions - if such classes can be defined. Circular yawing motion, for instance, is useful because it is a sort of approach to such a class. In fact, this application is achieved, perforce, in free-flight experiments. It should be noted, however, that the determination of the aerodynamic forces from the free-flight data is not a simple process; generally, it depends heavily upon an existence of a sufficiently idealized mathematical theory. Such a theory must start from a certain assumed force system, or, as it were, from a wind-tunnel point of view. It is hoped that this gap, as it were, between the possible free-flight technique and the complete non-linear Maple-Synge theory will be eventually bridged by extensive use of high-speed computing machinery, and by elaborate wind-tunnel instrumentation which would reproduce circular, elliptical, spiral, epicyclic, etc., yawing motion.

These vague generalities have cited here with the object of showing to the student of exterior ballistics the present groping state of the art: it is difficult not only to build a general non-linear theory, but even to formulate a general non-linear force system; and this even without considering asymmetries of the shell, liquidity of the filler, transitions from laminar flow, separation of the flow, intermittency of the turbulent wake, etc. Small wonder, then, that the linear theory remains a point of diminishing returns!

DAMPING TORQUE

Next to $\underline{M}$ in importance is the component of torque known as "damping" torque and designated by $\underline{H}$. Specifically, this torque is defined by an understanding that in direction it opposes the angular velocity $\underline{\eta}$ of the shell's axis, and in magnitude it depends essentially

on η. In the linear theory it is assumed to be proportional to η (i.e., to ζ), and is therefore mathematically analogous to viscous damping of a harmonic motion.* The choice of the term suggests that a hope must have been entertained that a torque so defined will account, in the main, for the damping of yawing motion. In fact, this is very nearly the case - although the damping of the yawing motion involves a number of phenomena other than H.

On Figure 9 H appears as an arrow directed from L to A_1; i.e., it tends to shrink the quasi-nutation LA_1 by pulling L toward A_1. Its effect upon the quasi-precession depends upon where T_1 is with respect to L and A_1; and in particular, in any circular yawing motion in which L is between T_1 and A_1, H tends to stretch the quasi-precession.

This simple formulation, however, has led to one of the pitfalls (mentioned in the introduction), which stunted for a while the development of our "non-mathematical" approach. For instance, in the linear theory precession and nutation are substantially circular yawing motions, at a constant ϕ/δ, i.e., with a constant proportion of quasi-nutation and quasi-precession. The theory shows that H damps the nutation, and causes a growth of precession. Thus in nutation H seems to cause a shrinkage of quasi-precession, and in precession H seems to cause a stretching of quasi-nutation - in a seeming contradiction to our simple formulation. In fact, a stimulating question was once asked - In a circular yawing motion, how does the torque H distinguish between nutation and precession?

These pitfalls are simply mistakes that are easy to fall into, but also easy to guard against. It is easy to forget that a shrinking or stretching of quasi-nutation and quasi-precession may be accomplished by the torque, M, as well as by H. It is also easy to assume that a yawing motion circular in the absence of H will remain, after H is introduced, a precession or a nutation (actually, it will develop into an epicycle).

In the presence of H the nutation and the precession are no longer circles, but spirals. In particular, in the linear theory we should look for those spirals which are "steady"; naturally, this means unchanging ratio ϕ/δ. This can indeed be the case if the torques shrink or stretch the quasi-precession in the same manner, proportionally, as the angular velocity η shrinks or stretches the yaw δ. Two such possible cases are shown in Figure 9. A shrinking spiral (SE regime) means L lagging behind the plane of yaw; then $\underline{M}$ has a component which shrinks the quasi-precession, working against the stretching action of H. An expanding spiral (NE regime) means L ahead of the plane of yaw, with a component of M stretching the quasi-nutation (against the shrinking action of H) as well as the quasi-precession.

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* However, H has nothing to do with the viscosity of the air.

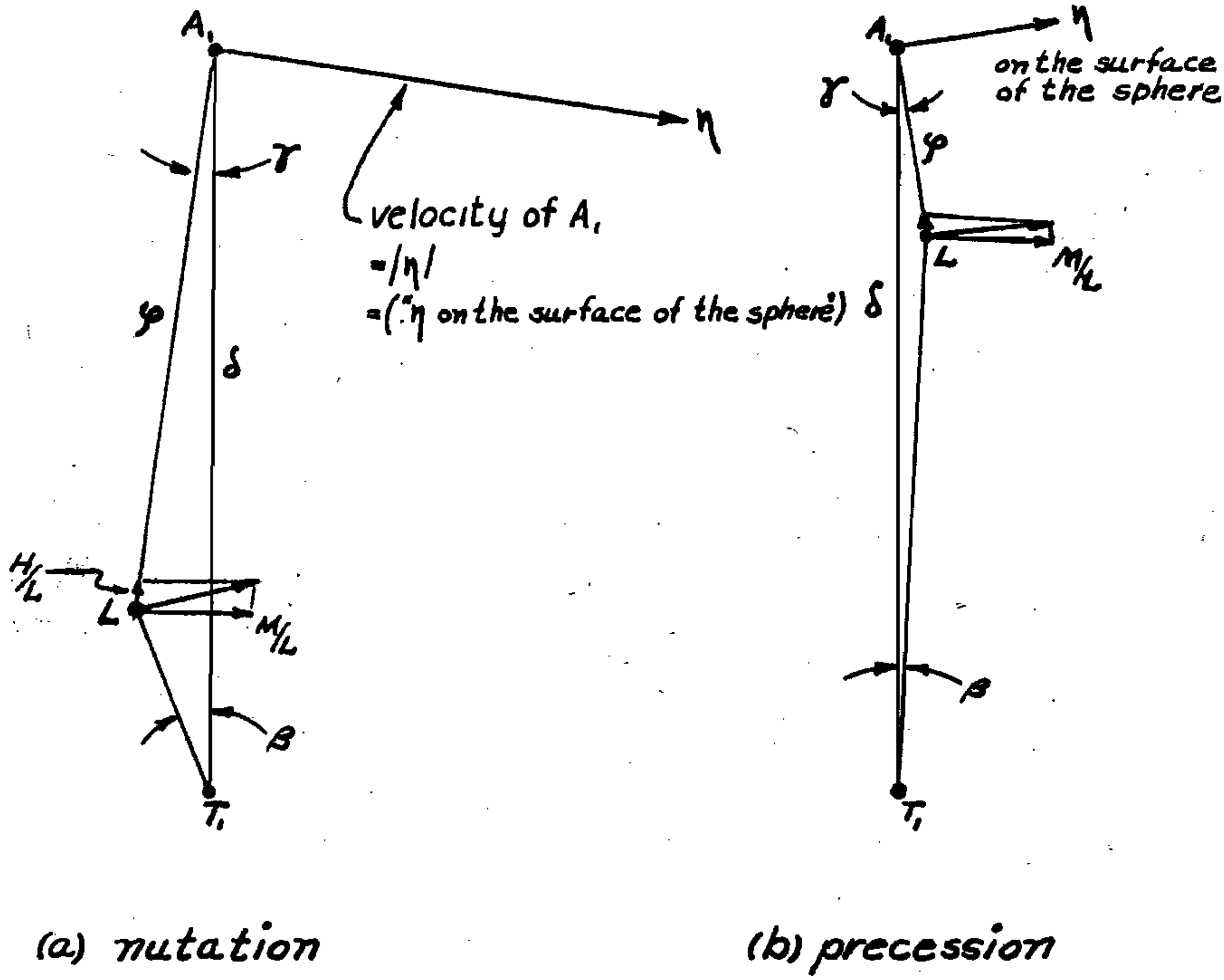

Fig 9: Effects of Damping Torque

Specifically, our requirement now is that the proportions of the triangle T_1LA_1 (rather than only the ratio ϕ/δ) remains constant; i.e., that the angular rates of both L and A_1 about T_1 be the same, and the angle between quasi-precession and the total torque be the same as the angle between the plane of yaw and the arrow of η. The spiral then, in particular, is a logarithmic spiral (this corresponds exactly to the situation in the mathematical approach, where we solve a differential equation by inquiring whether there may not be solutions the form $e^{\lambda t}$). In particular, we may assume that the angles LT_1A_1 and LA_1T_1 are small (the assumption of small torques, and hence small damping rates, is - together with the assumption of small yaw - a characteristic feature of linear theory). Then on Figure 9 we must have

$$(\pi/2 - \beta) + H/M = \pi/2 - \gamma \qquad (8)$$

Substituting $H = h\eta$, $\eta = (A\omega_1/B)\phi$, $M = \mu\delta$, and $\beta \cong \gamma\phi/(\delta-\phi)$ we may solve for γ in terms of the ratio ϕ/δ:

$$\gamma = h(A\omega_1/B\mu)(\phi/\delta)(1 - \phi/\delta)/(2\phi/\delta - 1) \qquad (8')$$

Since the triangles are very flat (spiral is much like a circle), we may take it that ϕ/δ is as given by (3B); then

$$\gamma = h(A\omega_1/4B\mu)(1 - \sigma^2)/\sigma \qquad (8'')$$

From the definition of σ in (3B) it may be readily noted that

$$1 - \sigma^2 = 4B\mu/A^2\omega_1^2,$$

so that upon substituting this into (8") and cancelling, we have the simple and curious result,

$$\gamma = \pm\, h/A\omega_1\sigma \qquad (8A)$$

i.e., the spirals of nutation and precession are mirror images of each other. However, since nutation goes around faster, it shrinks faster. Multiplying (8A) by the angular rates as given by (5), we have the proportional radial rates (the radial components of η divided by δ), with respect to time, as

$$\gamma\eta/\delta = \pm\,(h/A\omega_1\sigma)(A\omega_1/2B)(1\pm\sigma) = (h/2B)(1\pm 1/\sigma) \qquad (8B)$$

To compare this with the result of the standard ballistic texts (cf., particularly, BRL 446), we may substitute $h = K_H\rho d^4u$ and $B = mk^2d^2$; then the factor h/B is $(K_H\rho d^3/m)u/k^2d$, where the group of factors in the

parentheses is usually written as J_H, so that (8B) becomes $(J_H/2k^2)(u/d)(1\pm1/\sigma)$. The factor u/d drops out when we change the independent variable from time to the distance travelled by the shell and expressed in calibers. Thus, finally, the proportional radial rates are

$$(J_H/2k^2)(1\pm1/\sigma), \qquad (8C)$$

in agreement with the linear theory.

A curious student might inquire here whether some of our simplifying assumptions might not be rescinded at this point. Indeed, the recission of the assumption that the damping is small would be a straight-forward - if laborious - algebraic task; but the cumbersome result of this task would not be particularly useful, simply because this assumption is, generally, in excellent agreement with experiment. More important - and more difficult - would be a recission of the linearity. On a sphere there seems to be no clean-cut analog of the logarithmic spiral* - or, more exactly, it is hard to choose between many imperfect analogs. A crude sketch of several of such analogs is given in Fig. 10. We shall not inquire which ones of these analogs constitute "possible" yawing motions, and under which assumptions; it seems probable, for instance, that several of such-like motions might result under the identical assumptions about $M(\delta)$ and $H(\eta)$, if the initial conditions are suitably varied - in some vague analogy to the blending between the nutation and the precession in the non-linear regime. What matters for our present purposes is that the idea of nutation and precession - which in the non-linear regime has already lost its usefulness of superposition, and survived only for undamped circular motions - becomes even more complicated (viz., less useful) when damping is introduced. It is hoped that mathematicians will eventually clarify these concepts.

So far we have only retraced, verbosely and laboriously, some of the ground which is explored more swiftly and elegantly by mathematics; and we have refrained from doing that which we have, in effect, promised the reader: to defend the use, in ballistics, of that obvious mental process which - for want of a better name - we called "qualitative reasoning". Let us attempt it now. As soon as we broaden our approach by concerning ourselves with an approximate (as well as exact) prediction, the non-linearity ceases to be the bugaboo it was in any quantitative reasoning.

Let us consider a typical looping epicycle (linear or non-linear), with loops falling short of the origin. We sense that this is a case in which the quasi-precession predominates, somehow, over the quasi-nutation. The phase relation between A_1, L and T_1 alternates rapidly, and we surmise that the alternating stretching and shrinking of quasi-precession and quasi-nutation by M, on the whole, cancel; we may consider then only the direct effects of H.

* Since the sum of angles of a spherical triangle depends upon its size, there can be, generally, no similar triangles (as A_1LT_1) on the sphere.

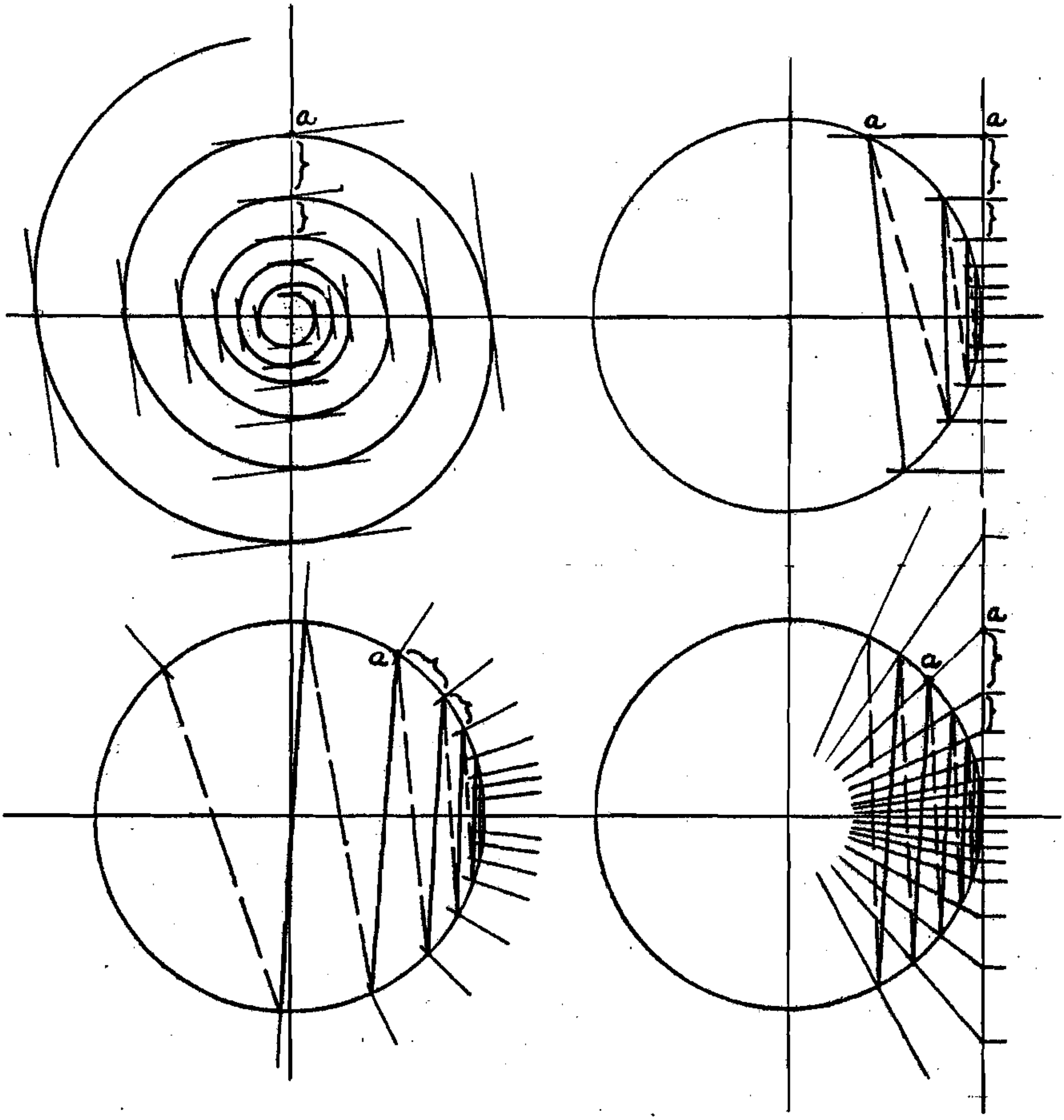

Fig 10: Examples of Possible Non-Linear Spirals Generated by a Given Linear Logarithmic Spiral

We infer that the quasi-nutation definitely damps. As to the quasi-precession, we observe that it is stretched when the angular velocity η is large (near the maximum of yaw), and shrunk when η is small (near the minimum of yaw); we infer that quasi-precession grows, but only slowly. We therefore infer that the long-range stability is present, with the following qualification. As the loops shrink, the shrinkage of quasi-nutation slows down, and the growth of quasi-precession becomes more steady; eventually, we no longer can assume that the phase relations of A_1 and L cancel out, but surmise that some steady phase relation, such as that of of Fig. 9b, begins to be felt, i.e., even the quasi-nutation will begin to grow; i.e., the long-range stability will eventually cease; although, by (8C), the growth of precession will be much slower than the shrinkage of the nutation had been (this circumstance is also obvious from a comparison of Fig. 9a and 9b: in precession H is smaller, and also it opposes, rather than adds, in some ways, the action of M). Now, this is the case even when we start with a motion in which the quasi-precession seems to predominate from the start. Had we started with an epicycle looping over the origin, the long-range stability would have been even more pronounced, since the situation would more often resemble Fig. 9a; yet we cannot claim the long-range stability for an indefinite range, as long as the situation is not precisely that of Fig. 9a - for a perturbation of the steady phase relations of Fig. 9a would tend to cause a growth of quasi-precession precisely at those times when η is greater.

Thus, our qualitative conclusion (without going into any mathematics) is that quasi-nutation damps strongly when it is a sizable component of yaw, grows slowly only when it is an unimportant component; the quasi-precession damps strongly when it is an unimportant component of yaw, grows slowly when it is the principal component of yaw. This conclusion falls pretty close to what is predicted by the exact linear theory; yet, it is completely free from the assumptions of linearity.

MAGNUS TORQUE

A positive Magnus torque* on all our sketches pulls L up. Thus in a general epicycle (if the oscillatory effects of M may be presumed to cancel out on the whole) T builds up the quasi-precession. The quasi-nutation is on the whole shrunk: for it is shrunk when δ and T are large, stretched only when δ and T are small. The steady spiral motions (Fig. 11) are possible if

$$(\pi/2 - \beta) + T/M = \pi/2 - \gamma \tag{9}$$

Substituting β and M as before, and $T = t\delta$, we can solve for γ:

$$\gamma = (t/\mu)(1 - \emptyset/\delta)/(2\emptyset/\delta - 1) \tag{9'}$$

* Magnus force is the force which deflects a spinning ball, and Magnus torque is the torque produced by the Magnus force. The reader is expected to be familiar with this (and other) historical background of Magnus effects.

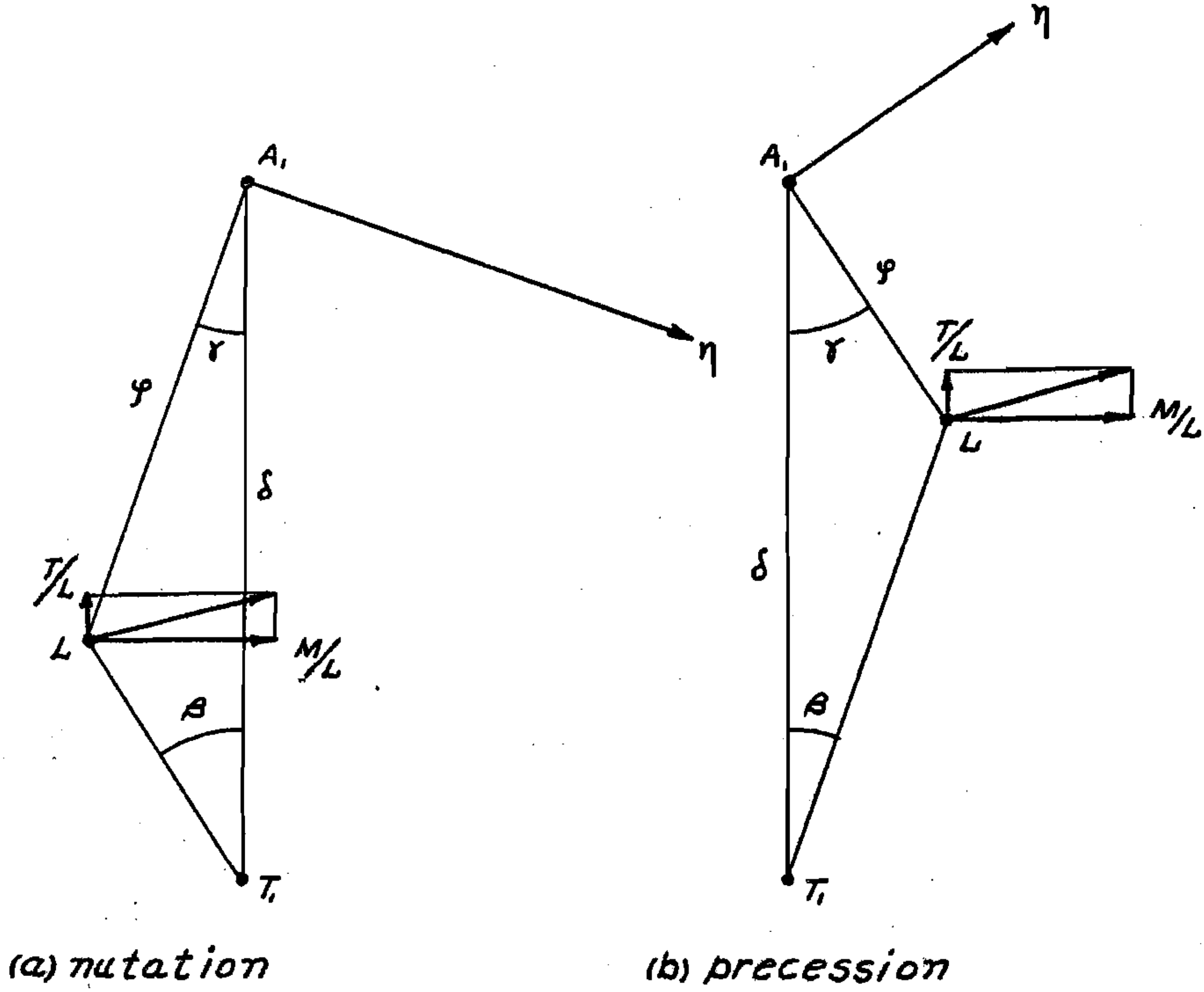

Fig 11: Effects of Positive Magnus Torque on Spin-Stabilized Shell

Using (3A) again, we have

$$\gamma = \pm (t/2\mu)(1 \pm \sigma)/\sigma = - (t/2\mu)(1 \mp 1/\sigma), \quad (9")$$

i.e., the two spirals are no longer mirror images: the precession spiral is steeper. Finally, substituting $\mu = A^2\omega_1{}^2(1 - \sigma^2)/4B$, $t = K_T\rho d^4 u\omega_1$ and $A = mk_a{}^2d^2$, and reverting to the notation of BRL 446, we have the proportional rates of growth of nutation and precession as

$$\mp (J_T/k_a{}^2)/\sigma, \quad (9C)$$

in agreement with the linear theory.

The effect of T is somewhat similar to that of H, the differences lying in the manner of dependence of damping upon the stability parameter σ, and also in the fact that T can be negative as well as positive. Many interesting examples of the interplay of these facts are discussed extensively elsewhere (cf., e.g., BRL 668 and BRLM 682). It should be noted that the study of the effects of T seems to be one occasion where it is desirable - because of the recently-discovered strong non-linearity of T - to substitute for the linearization (9') an exact, non-linear, solution of (9).

An illustration of the effects of H and T (as well as the effect of the aerodynamic forces, which we shall discuss presently) is provided by the spinning top. The friction at the tip of the top acts as some analog of aerodynamic cross-forces, and its torque as some analog of certain aerodynamic torques. The analogy is imperfect because the laws of Coulomb friction are not the same as those of the aerodynamic forces and torques (the friction force depends only upon the direction, rather than the magnitude, of the velocity at which the tip of the top rubs against the table). The greater complexity of the top's motion (in comparison with that of the shell) results in the apparent lively and capricious behaviour of the top; but there can be no doubt that some of the dispersion of artillery fire is akin to the wandering of the top.

At the start the center of gravity of the top is substantially stationary, while the tip rapidly swings out, sliding, substantially in a circle. The frictional force is substantially perpendicular to the plane of yaw, and acts as a positive Synge force S (or as a positive Magnus force F); it produces some analogs of H (or a positive T) - and also a slight spin-accelerating torque. As a result, the nutation damps out (rather abruptly), and presently only the pure precession remains. This precession may (because of H and positive T) grow for a while; but the damping is presently taken over by other processes. Being (in the slow precession) rather steady in its direction, the frictional force now causes some horizontal velocity of the c.g. across the plane of yaw: this tends to neutralize the rubbing velocity perpendicular to the plane of yaw, so that the tip tends to start rolling, rather than sliding, on the surface of the table. However, as the plane of yaw continues precessing, the rubbing velocity develops a component in the plane of yaw (the top starts, literally, to "draw" on the surface). The frictional force thus develops a

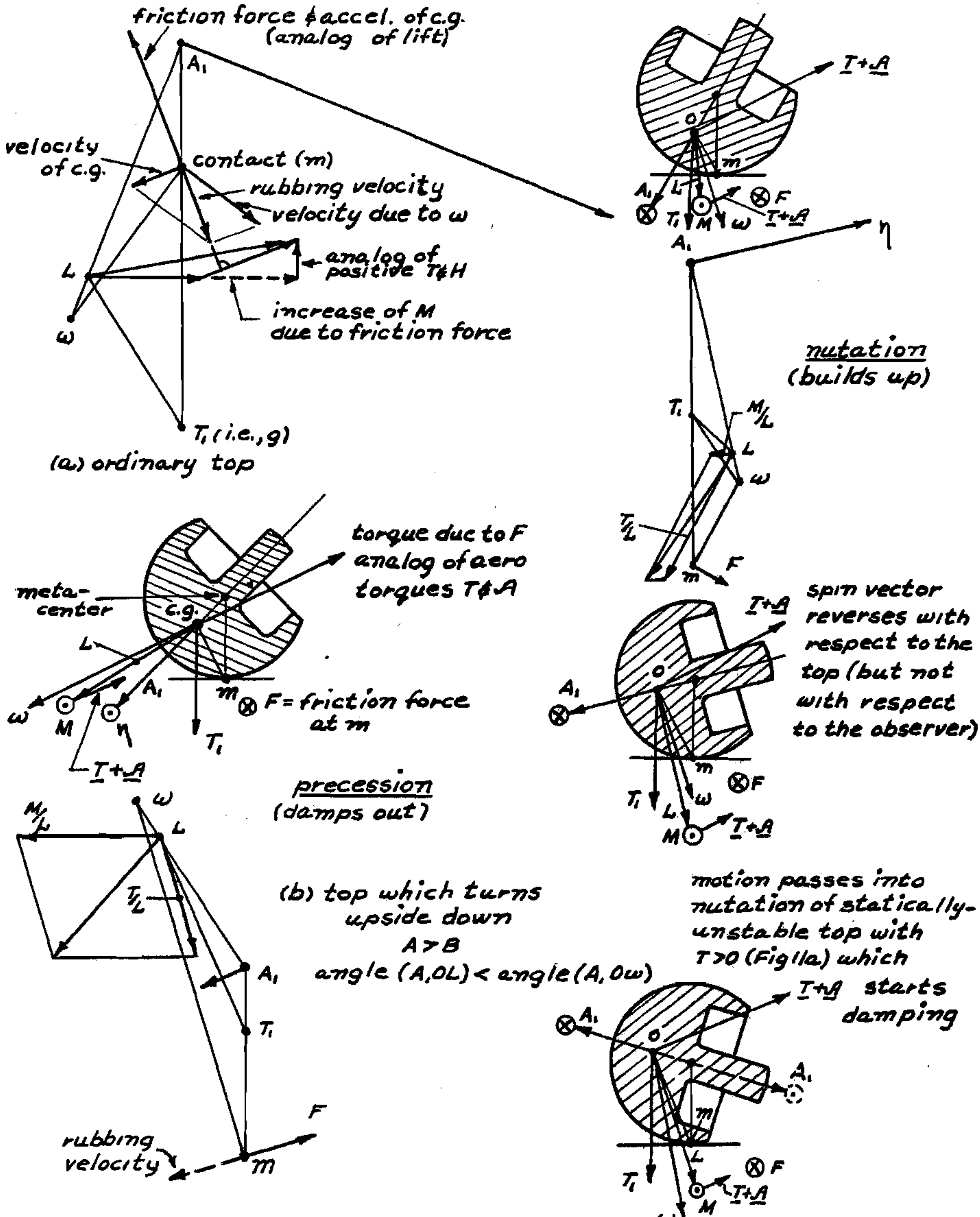

Fig 12: Friction in a Spinning Top as an Analog of Magnus Torque

component analogous to the lift L, which (as it will be presently shown) tends to damp the precession (there being no longer a nutation to grow). It seems to be an experimental fact (undoubtedly provable by analysis) that the effect of L overcomes that of H and T. This "dragging" neutralizes the velocity of the c.g., and finally the vector $\underline{\omega}$ comes within the radius of the tip; H (or T) become negative, and the top rather suddenly "goes to sleep".

Particularly interesting is the kind of the top which turns upside down. Such a top starts as statically stable: viz., the radius of its tip is large, and the metacenter* of the top is above the center of gravity (rather than at the tip). For the same reason the rubbing velocity due to spin is generally greater than the rubbing velocity due to yawing; the friction is analogous, therefore, to a negative Magnus torque. Now it is the precession which damps out rapidly, while the nutation builds up to $\pi/2$ and beyond. While the vector of angular velocity remains substantially vertical, it turns with respect to the top; for an instant the top is in the pure cartwheeling motion, without any axial spin. Thereafter the motion can be considered as the nutation of a statically-unstable top, with a positive Magnus torque - which presently commences damping. This is as though a fin-stabilized shell turned around and flew fins forward; in fact, the early stages of this motion are not unlike the motion of a mortar shell landing "short".

OTHER CROSS-TORQUES

The cross-torques M, H and T pretty well complete the cross-torque part of Kelley-McShane matrix (7); symmetry demands only that we mention the Magnus-cross-torque XT. This is one of the marginally relevant components, given rather an undue prominence by the Maple-Synge symmetry arguments; it now seems, for instance, that various non-linear components** of T are more important than XT (it should be noted, however, that XT has not been measured at really large yaws). On our sketches XT pulls L across the quasi-nutation; thus, in the general epicycle its damping effects rather cancel, and in circular yawing motions it is rather difficult (through not impossible***) to distinguish its effect from that of M, since it affects only the angular rates. This is in agreement with the theory of BRL 446, which shows that XT produces only a negligible effect on the angular rates.

We might remark on the apparent (through so far not established) checker-board and hierarchical character of this matrix. The linear M can be generalized into a function of both yaw and spin, odd in yaw and even in spin; all terms of such an expansion of M seem to affect only the angular rates, and leave damping unaffected. The torque T, odd in both yaw and spin (and perpendicular, by Maple-Synge theory, to M), appears as a sort of refinement on M; all its

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* Which is a point on the axis of the top (or fixed with respect to a ship) thru which the reaction of the table (or of water) acts, for small angles of yaw.

** Cf BRLM-682, "Spiral Yawing Motions of 81 mm Shell M56: a Study in Non-Linear Theory" (1953).

*** Cf BRL 882.

terms seem not to affect angular rates, but are very essential in damping. Now, the angular velocity η in itself is a sort of refinement on yaw δ; thus, H can be accounted for* by the variation of the instantaneous angle of attack along the axis of a yawing shell. Thus H, analogously to T, seems a refinement on M; it can be an even function of spin, does not affect the angular rates, and affects damping. Then XT appears a refinement on both H and T: it stands in the same relation to H as T to M, and to T as H to M; it is odd both in η and spin, affects angular rates and does not affect damping. This is as though the linear theory elevated XT on the principle of primogeniture, and disinherited its more important cousins, e.g., non-linearities of T.

CROSS-FORCES

The cross-forces - viz., the aerodynamic forces N, S, F and XF of matrix (7), as well as the component of gravity normal to the trajectory - affect the yawing motion by producing a swerve, i.e., by moving point T_1, or by shifting the base of the "vector" of quasi-precession, rather than that of the quasi-nutation. This is considerably different from the effect of the cross-torques, and our "qualitative reasoning", accordingly, will have to be considerably modified. Also, it will not be satisfying as it was with the cross-torques. The difficulty is quite real: we are dealing with a more complicated phenomenon than a mere angular motion, viz., with the effect of swerve upon the yawing motion. Mathematically, this is the step of coupling together two separate differential equations - a step laborious even in the linear theory. Indicative of this distinction is the obvious "cumulative" character of the effect of motion of T_1 upon the motion of L: the velocity of T_1 must have existed for some length of time, and in a substantially the same direction, before its effect is felt; or, the position of T_1 in the configuration of the triangle $T_1 LA_1$ reflects not the instantaneous velocity of T_1, but that velocity which had existed some time previously.

Lift. In advanced exterior-ballistics texts (e.g., BRL 446) the forces in the plane of yaw are resolved usually into the normal force N (perpendicular to the axis of the shell) and axial drag DA. It seems more natural to resolve them into lift $\mathcal{L}$ (perpendicular to the trajectory) and drag D (opposed to $\underline{T}_1$); in our case this is also more convenient.

On all our sketches lift pulls T_1 up. To the first approximation - if we can disregard the effect of M on the damping of the yawing motion - it is obvious that quasi-precession is shrunk by the lift (those rare occasions when L on our sketches is below T_1 represent minima of yaw, when lift is small). The effect of lift on the quasi-nutation is less obvious, and in fact is less important. To the first approximation we could simply say that a motion of T_1 has no effect upon L and A_1, i.e., has no effect on the quasi-nutation. Indeed, that would not be very far from the truth.

* Cf BRL 668; also, recent BRL monograph, "Exact Re-Statement of the Equations of Motion of a Shell".

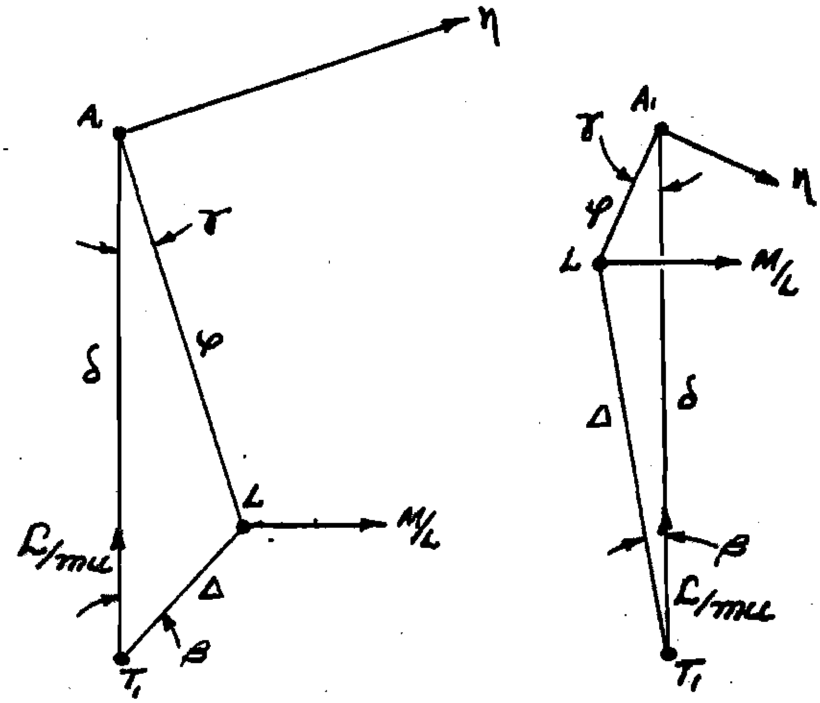
On a closer inspection we observe that the motion of T_1 does affect L by changing the yaw, and therefore changing the torque $\underline{M}$. Specifically, we may distinguish two effects of the motion of T_1.

The first one is as follows. In its continuous running toward A_1, the point T_1 fails, so to say, to "lead" A_1. The plane of yaw, therefore, turns more rapidly, and so does the arrow of M on our sketches. Inspecting our sketches (either one, or every one, according to the reader's patience) we observe that the faster turning of M tends to pull L away from A_1; we conclude that lift tends to stretch the quasi-nutation.

We are stepping now close to a pitfall. For instance, the linear theory shows that precession and nutation (under the assumptions which we have so far accepted, viz., a constant velocity and spin of the shell) have constant proportions ϕ/δ; but precession is damped by lift, while nutation is built up. Hence the quasi-precession in nutation is stretched, and quasi-nutation in precession is shrunk - in apparent contradiction to our preliminary observations. The answer lies, again, in the neglect of the effect of M, and in the unwarranted assumption that a motion representing a certain proportion of precession and nutation without the lift will retain this proportion after the lift is introduced. It is easy to confuse two different problems: one (ours), how does the lift affect a motion with given initial conditions; and the other (practically habitual in the milieu of mathematical approach), how does a possible spiral motion in the presence of lift differ from a possible spiral motion without lift.

We have an inkling of this in considering the second effect: as M turns, it also (on the whole) tends to diminish in magnitude, because of the generally diminishing yaw. The angular rate of quasi-precession, we might judge, is not affected much, for the magnitude of the quasi-precession also diminishes. But, since the velocity of A_1 is independent of the position of T_1, the angular rate of A_1 about T_1 is, on the whole, increased; i.e., the synchronism of L and A_1, or the spiral character of the motion, will not be generally preserved - unless the phase relations of the triangle $T_1 LA_1$ are such that the magnitude of quasi-nutation is changing in some particular manner. In this way we come to consider the spiral motions again. Two possible configurations are shown on Fig. 13, which explains the physical mechanism of the well-known result of the linear theory. Thus, in precession (Fig. 13b) the quasi-precession is shrunk not only by the lift, but also by the M-torque; the quasi-nutation is simultaneously strongly shrunk by the M-torque. In nutation (Fig. 13a) the M-torque stretches the quasi-precession more strongly that it is shrunk by the lift; and simultaneously, the M-torque slightly stretches the quasi-nutation.

Our qualitative conclusion (completely free from the assumptions of linearity) is, therefore, as follows. The quasi-precession is damped by lift strongly if it is an essential component of yaw, grows slightly if it is an unimportant component. The quasi-nutation grows slightly if it is the principal component of yaw, shrinks strongly if it is an unimportant component.



(a) nutation (b) precession

Fig 13: Effect of Lift

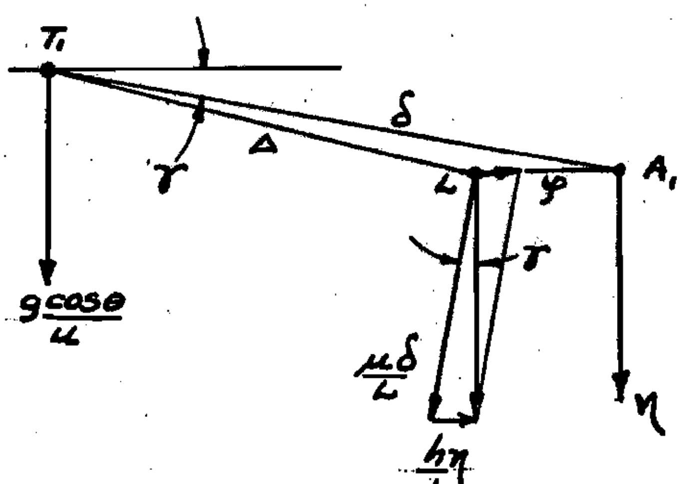


Fig 14: Yaw of Repose in the Presence of Damping Torque (an example)

The effect of lift should be juxtaposed with that of H-torque. In fact, these two refinements on an undamped epicyclic motion form a would-be satisfactory system for damping of yawing motion in the absence of Magnus effects: One damps nutation, the other - precession. This fact has been brought out particularly clearly by Davis and Follin*, to whose work I take pleasure to acknowledge my indebtedness once again. The student of ballistics should be reminded at this point that the Magnus effects had to be introduced, historically, precisely because that system failed to account satisfactorily for the totality of experimental information.

Let us now revert to the quantitative evaluation of the effect of lift in the linear case. Instead of considering the angles, as we did in (8) and (9), we should now consider (cf. Fig. 13) the proportional radial rates, i.e., the rates of growth divided by the magnitudes. To preserve similarity,

$$(\gamma\eta - \mathcal{L}/mu)/\delta = (\beta M/L - \mathcal{L}/mu)/(\delta - \phi) \qquad (10)$$

Substituting $\mathcal{L} = \ell\delta$, $M = \mu\delta$, $L = A\omega_1$, $\eta = (A\omega_1/B)\phi$, and $\beta = \gamma\phi/(\delta - \phi)$ and solving for γ, we have

$$\gamma = -(\ell/mu)(B/A\omega_1)(1 - \phi/\delta)/\left[(1 - \phi/\delta)^2 - B\mu/A^2\omega_1^2\right] \qquad (10')$$

Substituting, from (3") and (3B), $B\mu/A^2\omega_1^2 = (1 - \sigma^2)/4$ and requiring that the ratio ϕ/δ remain constant as in (3A), we have the simple result, somewhat analogous to (8A):

$$\gamma = \pm(\ell/mu)(B/A\omega_1)/\sigma \qquad (10A)$$

In the presence of swerve we no longer can compute the proportional radial rates (with respect to time) as $\eta\gamma/\delta$ - which we did in (8B); nor can we say - in analogy to (8A) - that the two spirals are mirror images of each other; rather, these rates are given by the left-hand side of (10). Substituting into (10) $\ell = K_L\rho d^2u^2$, and reverting to the notation of BRL 446, we have the proportional radial rates with respect to travel as

$$-(J_L/2)(1 \mp 1/\sigma), \qquad (10C)$$

in agreement with the linear theory.

There is some resemblance between (10C) and (8C), but it should not distract our attention from the fact that the mechanisms of the spirals of (10C) and (8C) are essentially different. The spirals of (8C) were yawing motions about a straight-line trajectory, those of (10C) represent an effect of swerve. Similarly, some further refinements on (10) are possible, but do not seem especially profitable.

* OSRD Report 2529, CIT JPC 29.

Magnus Force. While the appreciation of the Magnus effects arose originally from the knowledge of the existence of the Magnus force F (cf. the swerve of a spinning ball, Magnus's wind-tunnel experiments with spinning cylinders in cross-flow, Flettner's rotor, etc.), with a shell the most important effect of Magnus force F is that it produces the Magnus torque T - whose effects have already been discussed. Magnus force proper does produce some legitimate swerve*, (and in a spinning top, the friction at the tip produces some wandering of the top); but its direct effect on the yawing motion *per se* is, fortunately, negligible. A positive F on all our sketches pulls T_1 to the left; therefore, it affects mostly the angular rates (slightly increasing them), rather than the damping rates - and its effect is difficult to distinguish from that of a change in M**.

Synge Force. The force S which produces the damping torque H has been introduced into ballistics for the sake of a certain formal mathematical consistency. This force is closely analogous to Magnus force. On all our sketches this force pulls T_1 across the direction of the "vector" of quasi-nutation. Thus in the general epicyclic motion its effects rather cancel out (even more so than was the case with F), and in the spiral motions it affects only the angular rates (rather than the damping rates); it does have some effect on the swerve, but in the study of small yawing motion it can be disregarded.

Our qualitative conclusions about the unimportance of F and S are in agreement with the results of the linear theory.

Cross-Magnus Force. The force XF which produces the Magnus cross-torque XT needs mentioning only for the sake of completing the linear matrix (7); it survives in the linearization of the Maple-Synge theory only for the reason that Synge's aerodynamic hypothesis treats the yawing velocity η as being on par with the yaw δ (rather than as a refinement on δ, as we hold). In the results of the linear theory of yawing motion the effect of this force is apparently negligible. Yet we may surmise that the cross-force part of the matrix (7) may possess the same checker-board character which we have detected in the cross-torque part of that matrix; then XF should affect damping. On our sketches XF pulls T_1 along the direction of quasi-nutation; so that, indeed, it affects damping in the basic, spiral, motions. It is a simple matter to investigate the effect of XF on spiral motions quanitatively, with the assumptions of linearity - analogously to (10). The result is that the radial proportional rates of growth with respect to the distance in calibers are

$$\pm J_{XF} J_M / k_a^2 \sigma \qquad (11C)$$

* Cf., e.g., BRL 703, "On Jump Due to Muzzle Disturbances"; also BRLM 682.

** An evaluation of the effect of F on the angular rates could readily be made, but does not appear profitable: for, with the assumptions usually made in the linear theory, it turns out that this effect is negligible.

This is in agreement with what the result of linear theory would have been if terms of second order in J's had been kept. Thus, XF does not matter in yawing motion simply because its effects are small. We may note that our surmise about the checker-board character of (7) is confirmed, but is overwhelmed by the hierarchical character of that matrix. If the reader will tolerate another figure of speech, the primogeniture claims of XF, filed with the Maple-Synge theory, are rejected even by the Kelley-McShane theory which is descended from Nielsen-Maple-Synge concepts.

The checker-board character of (7) can be summed up by writing

$$\begin{matrix} M(a) & T(d) \\ H(d) & XT(a) \end{matrix} \quad \text{and} \quad \begin{matrix} N(d) & F(a) \\ S(a) & XF(d), \end{matrix}$$

where (a) and (d) stands for "affects angular rates" and "affects damping rates". The hierarchical character of (7) can be summed up by ranking its components in the order of their importance:

1st rank	M	(a)	determines the basic character of motion
2nd "	H, T, N	(d)	determine damping
3rd "	XT, F, S	(a)	refinement on angular rates
4th "	XF	(d)	refinement on damping rates.

The usefulness of this trivial classification lies in the fact that it shows up certain weaknesses of Nielsen-Synge-Kelley-McShane linearization of Maple-Synge theory, and helps to organize the methods of further attack: thus, if we study the damping rates, no direct effect of XT, F and S can be expected, while refinements on H, T and N are just as legitimate - and may be more important - than the measurement of the "first-order" term XF.

YAW OF REPOSE

The effect of the force of gravity acting across the trajectory, which causes the curvature (as distinguished from the swerve) of the trajectory, is considerably different from that of the aerodynamic cross-forces - for the simple reason that this force is relatively constant and independent of yaw and yawing velocity. Mathematically this force manifests itself an inhomogeneity of the differential equation (rather by a change in the coefficients of the equation) and results in the fact that the general (linear) solution consisting of precession and nutation must be augmented by the third component, called yaw of repose.

We may arrive at the same conclusion by an argument analogous to that by which we have introduced the generalized (non-linear) precession and nutation. The point T_1 "falls", in the surface of our sphere, (cf. Fig. 14) at the velocity $g\cos\theta/u$ (where g is the acceleration of the gravity and θ is the angle of inclination of the trajectory). If the shell is to trail properly, the vectorial velocity of A_1 on this sphere must be approximately the same. Dropping the convention of drawing A_1 above T_1, we observe that L must be directly

to the left of A_1, i.e., there must exist a horizontal quasi-nutation, pointing (from L to A_1) to the right. Moreover, the point L of this sphere must have substantially the same vectorial velocity, i.e., there must exist suitable torques; we assume these to consist largely of the M-torque. Then there must exist a yaw (from T_1 to A_1) pointing directly to the right. Thus T_1, L and A_1 are co-planar and on our sketches the situation appears as a circular yaw, except that the motion of T_1 causes the triplet of these points to move down without any rotation noticeable in our sketches.

The condition of the "steadiness" of this motion is

$$g\cos\theta/u = M(\delta)/L = \Omega\sin\phi \qquad (12)$$

which, on assumptions of linearity, becomes simply

$$g\cos\theta/u = \mu\delta/A\omega_1 = (A\omega_1/B)\phi, \qquad (12')$$

and from which there follow the simple results

$$\phi/\delta = 1/4s \qquad (12")$$

and

$$\delta = (A\omega_1/\mu)(g\cos\theta/u) \qquad (12A)$$

The expression (12A) for the yaw of repose may be rewritten in a number of ways. For instance, in the notation similar to that of BRL 446 and BRL 668, we would introduce the dimensionless quantities $\nu = \omega_1 d/u$ = spin per caliber of travel, and $J_{\theta c} = gd\cos\theta/u^2$; then (12A) becomes

$$k_a^2 \nu J_{\theta c}/J_M, \qquad (12C)$$

in agreement with the skeletonized form for the yaw of repose in the linear theory.

The yaw of repose is considered, naturally, a basic form of yawing motion, analogous to the precession and the nutation of (3B). Its superposability with the nutation and the precession is, of course, distinctly a feature of the linear theory. In attempting to combine the yaw of repose with the epicycle in the general non-linear theory we would meet, unfortunately, the difficulties analogous to those we met in attempting to pass from the linear to non-linear epicycle. A study of the non-linearities of the yaw of repose existing alone (analogous to our review of the theory of the spinning top) does not appear profitable.

It is a simple matter, however, to generalize (12), within the assumptions of linear theory, by perturbing it for the effect of other forces and torques. At this stage we may readily make the perturbations for the cross-torques and cross-forces: these would be very similar to those of (8), (9), (10) and (11),

except that the conditions of spirality would be replaced by the conditions of a parallel synchronous motion, e.g., (3B) replaced by (12"). Just to indicate the simplicity of such processes, we might make the perturbation of (12') for the H-torque. From Fig. 14 it is obvious that we must have

$$\mu\delta\gamma = h\eta \tag{13}$$

from which, using (12"),

$$\gamma = h/A\omega_1 \tag{13A}$$

and the downward component of the yaw of repose can be shown to be

$$\delta\gamma = J_{\theta c}J_H/J_M, \tag{13C}$$

in agreement with the linear theory.

Additional perturbations on the yaw of repose (including the perturbations for the axial forces and torques, which we have not made as yet for precession and nutation) we shall leave as an exercise for the reader: and we need hardly mention that our method can reproduce all the results of the linear theory. We might only remark that in any respectable trajectory the yaw of repose is extremely small, and is, therefore, in itself only a minor refinement on the epicycle. Experience shows, in some cases, that an apparently large summital yaw is very likely due to a survival of the epicyclic yaw, i.e., insufficient damping. We might also mention that $J_{\theta c}$, unlike the aerodynamic J's, is proportional to the caliber - so that the yaw of repose must be scaled from one caliber to another by other rules than those of the epicycle*.

FORCES ALONG THE TRAJECTORY

Let us now consider the effects of drag (D) and of the component of gravity (mgsinθ) acting in the direction opposite to the trajectory vector $\underline{T}_1$. Their effects differ considerably from those of the cross-forces (just as the "WKB" perturbation of BRL 446 differs from the inclusion of the complex components into the coefficients of a differential equation). They affect the yawing motion by changing the velocity of the shell, and hence changing the overturning moment $\underline{M}$ and the stability factor s. In precession and nutation - cf. (3B) and (4B) - the proportions of the spherical triangle T_1LA_1 are no longer constant, even in the linear theory.

Let us start these considerations first in the most general (and that necessarily means vague) way, particularizing them presently by resorting to the spiral motions, and finally, to the assumptions of linearity. Let us start at the instant of minimum yaw (either W or E sketch). We note that throughout the regimes of increasing yaw (NW, N and NE sketches) the motion of the point L, caused by $\underline{M}$, contributes to the increase of yaw - specifically,

* Cf. BRLM No. 833, "Trajectory Models in Mortar Fire".

by tending to stretch both the quasi-precession and the quasi-nutation. Hence the decrease of $\underline{M}$, caused by the loss of velocity, decreases the growth of yaw.

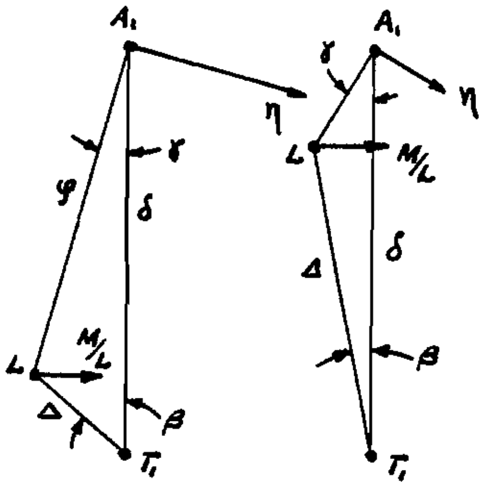
In the regimes of the decreasing yaw (SE, S and SW sketches) the motion of L tends to shrink both the quasi-precession and the quasi-nutation. This fact, taken alone, does not give us the information as the total effect on yaw: both the quasi-precession and the quasi-nutation are decreased by the loss of velocity of the shell; but the minimum yaw is the difference between the two. We can observe then that in these regimes the motion of L impedes the shrinkage of yaw. Hence the decrease of the speed of L - by decreasing the impeding of the shrinkage of yaw - increases the shrinkage.

We conclude tentatively that the loss of velocity always damps the yaw of a spin-stabilized projectile.

We may note that this damping is associated with an increase of the stability factor; and it is interesting that our intuitive sense of the existence of a relation between the stability and the amplitude (or damping) of the yaw is, in a way, confirmed.* We may also note that at this stage we are not able as yet to make a definite statement about the shrinkage of the quasi-precession and the quasi-nutation; this corresponds to the fact that we made no committment about the shape of the epicycle, viz., about the proportions of the precession and nutation (or quasi-precession and quasi-nutation) in this epicycle. It may be seen, for instance, from (3B) that different behavior of the quasi-precession and quasi-nutation may be expected, depending upon whether the nutation or the precession predominates.

We arrive at the consideration of spiral motions by a process somewhat similar to that we used in discussing the effects of lift. The angular velocity Ω of quasi-nutation about the instantaneous position of point L remains constant, but the linear velocity of point L in our sphere decreases: thus a possible synchronism of the quasi-precession with the plane of yaw tends to be disrupted. Yet this synchronism may be preserved, approximately, if the magnitude of the quasi-precession decreases, too. This can be done by the torque $\underline{M}$ if L lags the plane of yaw; the situation must be then described by a SE sketch, and the quasi-nutation must shrink, too. But the spiral motions must now be considered in conjunction with the fact that the ratio ϕ/δ from (3B) or (4B) is no longer a constant; i.e., the quasi-nutation and quasi-precession are decreasing at different proportional rates, the angles β and γ in the triangle T_1LA_1 are changing, and the synchronism of the quasi-precession with the plane of yaw can no longer be exact. Specifically, in nutation (Fig. 15a) the quasi-precession will shrink more rapidly, and the

* With fin-stabilized projectiles the loss of velocity builds up the yaw - but there the loss of velocity makes the shell more sluggish in its yawing motion, or - in our intuitive sense - less stable.



(a) nutation (b) precession

$$In\ (1-\sigma^2) = \frac{1}{S} = \frac{4BK_M\rho d^3 u^2}{A^2\omega_1^2}$$

only u changes; hence

$$-2\sigma\dot{\sigma}_u = \frac{2\dot{u}}{Su}$$

$$\dot{\sigma}_u = -\frac{1}{S\sigma}\cdot\frac{\dot{u}}{u}$$

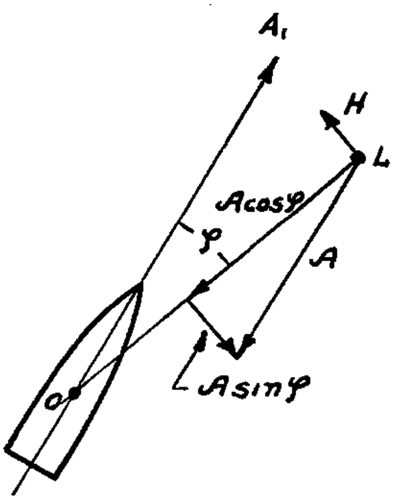
$$\frac{\dot{\sigma}_u}{2\sigma} = -\frac{1}{2S\sigma^2}\cdot\frac{\dot{u}}{u} = \frac{(1-1/\sigma^2)}{2}\cdot\frac{\dot{u}}{u}$$

But $\dot{u} = -\dfrac{K_D\rho d^3u^2}{m} - g\sin\theta$

$$\therefore \quad \frac{\dot{u}}{u} = -(J_D+J_\theta)\cdot\frac{u}{d}$$

$$\therefore \quad -\frac{\dot{\sigma}}{2\sigma} = \frac{(J_D+J_\theta)}{2}(1-1/\sigma^2)\cdot\frac{u}{d}$$

Fig 15: Effects of Drag



analogy

$$H = K_H\rho d^4 u\,\frac{A}{B}\,\omega_1\varphi \longrightarrow K_A\rho d^4 u\,\omega_1\varphi$$

$$-\frac{J_H}{2k^2}\left(1\pm\frac{1}{\sigma}\right) \longrightarrow +\frac{J_A}{2k_a^2}\left(1\pm\frac{1}{\sigma}\right)$$

If $In\ (1-\sigma^2) = \dfrac{1}{S} = \dfrac{4Bu}{A^2\omega_1^2}$ only ω_1 changes

Then: $\quad 2\sigma\dot{\sigma}_\omega = \dfrac{2\dot{\omega}_1}{S\omega_1}$

And: $\quad \dot{\sigma}_\omega = \dfrac{1}{S\sigma}\cdot\dfrac{\dot{\omega}_1}{\omega_1}\ ; \quad \dfrac{\dot{\sigma}_\omega}{2\sigma} = \dfrac{1}{2S\sigma^2}\cdot\dfrac{\dot{\omega}_1}{\omega_1} = -\dfrac{(1-1/\sigma^2)}{2}\cdot\dfrac{\dot{\omega}_1}{\omega_1}$

$$\frac{\dot{u}}{u} = -(J_D+J_\theta)\,\frac{u}{d} \longrightarrow \frac{\dot{\omega}_1}{\omega_1} = -\frac{K_A\rho d^4 u\,\omega_1}{mk_a^2d^2\omega_1} = \frac{J_A}{k_a^2}\cdot\frac{u}{d}$$

$$(J_D+J_\theta)\,\frac{(1-1/\sigma^2)}{2} \longrightarrow -\frac{J_A}{k_a^2}\,\frac{(1-1/\sigma^2)}{2}$$

$$\frac{J_A}{2k_a^2}\left[(1\pm 1/\sigma)-(1-1/\sigma^2)\right] = \frac{J_A}{2k_a^2}\left(\pm 1/\sigma + 1/\sigma^2\right)$$

Fig 16: Effects of Axial Torque

ratio ϕ/δ will increase, as it should by (3B); the angle β will increase - and in the limit, when stability factor becomes infinite, σ approaches 1, the motion becomes the vacuum nutation, and points L and T_1 merge, β ceases to have meaning. Analogously, in precession (Fig. 15b) it is the quasi-nutation which shrinks more rapidly, the ratio ϕ/δ decreases, and so does the angle β (and in the limit it is ψ which ceases to have meaning).

Passing to the quantitative evaluation of the rates of shrinkage on the assumptions of linearity, we must now drop our pretence of avoiding calculus: we are now concerned with the rates (i.e., derivatives) more intimately, and we are parallelling a more advanced mathematical process (the WKB perturbation for the non-constancy of the coefficients of the differential equations) than before. Still, we shall need not the theory of the differential equations, but only the most elementary operations of the differential calculus. If we let Δ be the quasi-precession $(\delta - \phi)$, we can re-write (3B) as

$$\Delta/\delta = (\delta - \phi)/\delta = 1 - \phi/\delta = (1 \mp \sigma)/2 \qquad (3C)$$

From Fig. 15 we observe that the rate of change of Δ is $\dot{\Delta} = -\beta(M/L)$, and the rate of change of δ is $\dot{\delta} = -\gamma\eta$; it will be an easy matter to relate the change of velocity to the rate of change $\dot{\sigma}$ of σ. To relate these rates ($\dot{\Delta}$, $\dot{\delta}$ and $\dot{\sigma}$) we need only to differentiate (3C):

$$\dot{\Delta}/\delta - \Delta\dot{\delta}/\delta^2 = \mp \dot{\sigma}/2 \qquad (14)$$

which, we might note, we do not consider as a differential equation. Substituting the expressions for $\dot{\Delta}$ and $\dot{\delta}$, and further substituting $\beta = \gamma\phi/\Delta$ and $\eta = (A\omega_1/B)\phi$ as before, we have the equation for γ, from which

$$\gamma = \pm\, \sigma/2(\phi\mu/\Delta A\omega_1 - A\omega_1\phi\Delta/\delta^2), \qquad (14')$$

Substituting into (14') the values of ϕ/δ and Δ/δ from (3B) and (3C), and simplifying as before, we have

$$\gamma = -\dot{\sigma}/2\sigma(A\omega_1/2B)(1 \pm \sigma) \qquad (14A)$$

Obviously, the proportional rates of change $(\dot{\delta}/\delta)$ are

$$-\dot{\sigma}/2\sigma \qquad (14B)$$

for both precession and nutation; and little algebra need be done to show that the radial rates with respect to travel are

$$+(J_D + J_\theta)(1 - 1/\sigma^2)/2 \qquad (14C)$$

(where $J_\theta = gd\sin\theta/u^2$), in agreement with the linear theory.

Of course, the mechanism of the resultant logarithmic spirals is again different from that of the spirals of (8C) and 10C). It is indeed a tribute to the elegance of the mathematical linear theory that three such distinct physical processes result in the same simple (exponential) type of solution.

It may be remarked that the subscript D in (14C) refers to the full drag (rather than the axial drag DA), i.e., includes the yaw-drag; while in the standard texts on linear theory J_D is really meant as J_{DA}. A more thorough (up to a point) study of the theory indicates, indeed, that it is more nearly correct to use J_D (in our sense) than J_{DA} - as the student will, no doubt, have occasion to find out.

AXIAL TORQUE

It now remains only to inspect the effect of the torque ($\mathcal{A}$) acting along the axis of the projectile. In the linear theory it is the spin-decelerating torque, $= K_A \rho d^3 u^2 \nu$, and its effect is rather completely overshadowed by the increase of stability due to the loss of velocity. This torque can be readily generalized, by the Maple-Synge theory, to include terms analogous to yaw-drag, etc.*

Here, at last, is one torque that truly maps on our sketches as a point - viz., the point A_1. It was mentioned, however, that in our method torques must be decomposed along and across $\underline{L}$. In the foregoing discussion of the cross-torques - generally, following the assumptions of linearity - we had neglected this subtle distinction, treating torques normal to $\underline{A}_1$ as though they were normal to $\underline{L}$: i.e., we had in effect assumed $\cos\phi = 1$, or our error was of the second order in ϕ. The component of $\mathcal{A}$ along $\underline{L}$ is approximately same as $\mathcal{A}$, but its component across $\underline{L}$ is approximately $\bar{\mathcal{A}}\,\phi$, viz., of the first order in ϕ (Fig. 16). The first component, insofar as our sketches are concerned, acts analogously to a negative drag, i.e., by changing σ (except that the change of σ due to change of ω_1 must be distinguished from the change of σ due to change of u)**; the second component is readily seen to be analogous to the H torque (except for the sign and a constant factor). Combining the effects of these two components, it can be readily shown that the proportional radial rates with respect to travel are

$$(J_A/2k_a^2)(\pm 1/\sigma + 1/\sigma^2), \tag{15C}$$

in agreement with the linear theory.

Again, certain refinements by the recission of the assumptions of linearity are possible, if not profitable. In particular, it will be readily seen that there are some effects due to the components of cross-torques along $\underline{L}$.

---

* Cf. also BRL 668 and BRL 882.

** Not to be confused with the fact that the arrow of $\mathcal{A}$ on Fig. 16 is in the direction of positive axial drag.

$\frac{2r}{1-a}$	rate of nutation with respect to precession / rate of precession
-2	
0	
2/3	
1	
2	
3	
8	
$\pm\infty$	
-6	
-3	
-2	

CONCLUSION

We have thus accomplished the following:

1. Established an interpretation, and a visualization, of the well-known results of the mathematical linear theory of yawing motion. This visualization is not limited to the linear theory, and may be used as a starting point for the more advanced problems of ballistics.

2. Reproduced all results of the linear theory without resorting to calculus (saving that excellent tool for the more worthwhile problems of ballistics).

3. Reviewed the theory of yawing motion in the light of the latest experimental information (cf., e.g., the importance of the non-linearities of T in comparison with XF).

4. Outlined (crudely) effects of some of the non-linearities of ballistics.

5. Drawn attention to a certain inherent weakness of the present-day mathematical approach, viz., the sharp split between the simple linear, and generally-unsolved non-linear, cases. It is hoped that some day someone will introduce some radical solution of this difficulty; it might take the form, for instance, of a special-purpose analog differential analyzer that has, so to say, the spherical trigonometry built into it.

The justice of the probable criticisms, that (a) we could not have done this work had the mathematical approach not blazed the trail, and (b) it would be simpler to learn calculus, - is so obvious that it requires no comment.

Incidentally, our approach does seem to have blazed the trail in two more advanced problems: the effect of asymmetry of the shell, (as mentioned on page 13 of BRLM 685) and the effect of the liquidity of the filler. However, both of those problems are outside the scope of this paper.

Serge J. Zaroodny

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REMARKS ON NOTATIONS

Effort was made to adhere to familiar notation, even at the cost of some minor inconsistencies.

Underlined letters stand for three-dimensional vectors. Otherwise letters are used for: scalar quantities (e.g., magnitude of the vector); to designate a point on the sketch; and on a few occasions, explained in the context, for complex quantities.

Subscript 1 is used to designate a vector of unit magnitude; an exception is the axial spin ω_1, where this subscript is used (after BRL 446) for the axial component of the angular velocity $\underline{\omega}$; the cross-component is considered either as the vector $\underline{\eta}$, or as the ~~scalar~~ η, but is visualized as an arrow on the sketch, viz., essentially as the complex quantity ζ of BRL 446.

While the dot over a letter indicates, as usual, the derivative of a quantity with respect to time, these derivatives (spoken of as "rates") are introduced each time as separate symbols.

Subscripts n and p refer to nutation and precession.

Consistent units are assumed throughout. All angles are in radians (though often units of angles do not matter).

m = mass of shell; d = caliber; u = velocity; g = gravity; θ = angle of inclination of trajectory, reckoned positive upward; ρ = density of the air; A and B are axial and transverse moments of inertia, and k_a and k are the corresponding radii of gyration in calibers; ν = spin in radians per caliber of travel; s = stability factor, as defined by (3*).

$\sigma = \sqrt{1 - 1/s}$, the very convenient and familiar (though still nameless) parameter of exterior ballistics, particularly in the linear theory; its physical significance is clear from (3B).

$\underline{T}_1$ = direction of trajectory (velocity of the c.g. of shell); $\underline{A}_1$ = direction of shell's axis; $\underline{L}$ = angular momentum of the shell; $\underline{G}$ = total torque acting upon the shell. Letter O on some sketches denotes shell's center of gravity.

δ = yaw (angle from $\underline{T}_1$ to $\underline{A}_1$); ϕ = quasi-nutation (angle from $\underline{L}$ to $\underline{A}_1$; also denoted by QN); Δ = quasi-precession (angle from $\underline{T}_1$ to $\underline{L}$; also denoted by QP); Ω = angular velocity of quasi-nutation with respect to the instantaneous position of $\underline{L}$, cf. (2); $f(\delta,\phi)$ = the trigonometric function defined by (4) and relevant for instantaneously circular yawing motion: cf. Fig 6; γ and β are auxiliary angles, as denoted on the sketches.

M = overturning moment; H = damping moment; T = Magnus moment; XT = cross-Magnus moment; N = Normal force; F = Magnus force; S = Synge force; XF = cross-Magnus force; D = drag; DA = axial drag. Italics $\mathcal{A}$ and $\mathcal{L}$ are used to distinguish the axial torque and lift from the moment of inertia and angular momentum; but little confusion would arise if A and L are used, as is customary. After Synge, the words "moment" and "torque" are used interchangeably, as are "angular velocity" and "spin". K's are defined in BRL 446; similarly, $J = K\rho d^3/m$; $J_\theta = gd\sin\theta/u^2$ and $J_{\theta c} = gd\cos\theta/u^2$. As is customary, coefficient μ is associated with M; to complete the symmetry (and in minor variance from Fowler and BRL 664), h is analogously associated with H, t with T, and ℓ with L.

Points of compass (N, NE, E, SE, S, SW, W, NW) are used to indicate the direction of the yawing motion, as viewed on Fig. 4, qualitatively.

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